An Introduction to Supersymmetry

Voja Radovanović
Faculty of Physics
University of Belgrade

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1 Notation and conventions

1.1 Lorentz and $SL(2,C)$ symmetries

In these lectures we use 'mostly minus metric', i.e. $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. Lorentz transformations leave metric unchanged, $\Lambda^T g \Lambda = g$. The proper orthochronous Lorentz transformations, $L_+^\uparrow$, consisting of boosts and rotations, satisfy the following two conditions:

$$\det \Lambda = 1, \quad {\Lambda^0}_0 \geq 1 \; . \tag{1}$$

These transformations are a subgroup of Lorentz group. Lorentz algebra has six generators, $M_{\mu\nu} = -M_{\nu\mu}$. The commutation relations are

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho}) \; .$$

The generators of rotations are $J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$, while generators corresponding to the boosts are $N_k = M_{0k}$. Further, one can introduce the following complex combinations $A_i = \frac{1}{2}(J_i + iN_i)$ and $B_i = \frac{1}{2}(J_i - iN_i)$. It is easy to prove that

$$[A_i, A_j] = i\epsilon_{ijl}A_l, \quad [B_i, B_j] = i\epsilon_{ijl}B_l, \quad [A_i, B_j] = 0 \; .$$

This is a well–known result which gives a connection between the Lorentz algebra and "two" SU(2) algebras. Irreducible representations (i. e. fields) of the Lorentz group are classified by two quantum numbers (j_1, j_2) which come from above two SU(2) groups. $L_+^\uparrow$ is double connected (due to $SO(3)$ subgroup) and non-compact group.

To discuss its representations one has to consider its universally covering group, $SL(2,C)$. $SL(2,C)$ is the group of 2×2 complex matrices with unit determinant. Pauli matrices are given by

$$\sigma^\mu = (1, \boldsymbol{\sigma}), \; \bar{\sigma}^\mu = (1, -\boldsymbol{\sigma}) \; ,$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; .$$

To find the connection between $SL(2,C)$ and $L_+^\uparrow$ we define one-to-one map between Minkowski coordinates, x^μ and 2×2 Hermitian matrices:

$$X = x_\mu \sigma^\mu = X^\dagger = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_0 + ix_2 & x_0 - x_3 \end{pmatrix} \ . \tag{2}$$

Let us now transform X to another Hermitian matrix in the following way $X \to X' = MXM^\dagger$, where $M \in SL(2,C)$. This transformation preserves the interval, i.e.

$$\det X' = \det X \Leftrightarrow x'^2 = x^2.$$

From

$$x'^\mu = \frac{1}{2}\mathrm{Tr}(\bar\sigma^\mu X') = \frac{1}{2}\mathrm{Tr}\Big(\bar\sigma^\mu M \sigma_\nu M^\dagger\Big) x^\nu = \Lambda^\mu{}_\nu x^\nu, \tag{3}$$

it follows that

$$\Lambda^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}\Big(\bar\sigma^\mu M \sigma_\nu M^\dagger\Big).$$

The mapping $M \to \Lambda(M)$ is $2-1$ homomorphism between $SL(2,C)$ and $L_+^\uparrow$. This mapping is not an isomorphism since matrices $\pm M$ correspond to Lorentz matrix Λ. $SL(2,C)$ is simply connected, so it is the universal covering group of $L_+^\uparrow$.

1.2 Representations

A left-handed Weyl spinor, $\psi_\alpha(x)$ is two component object, which transforms in the following way

$$\psi'_\alpha(\Lambda x) = M_\alpha{}^\beta \psi_\beta(x) \tag{4}$$

under $SL(2,C)$ transformations. The matrix M belongs $SL(2,C)$ group. Indices α and β take values $1,2$. In Quantum Field Theory $\psi_\alpha(x)$ is an operator. Its transformation law under Lorentz transformations is

$$U^{-1}(\Lambda)\psi_\alpha(x)U(\Lambda) = M_\alpha{}^\beta \psi_\beta(\Lambda^{-1}x) \ . \tag{5}$$

Let us define $\bar\psi_{\dot\alpha} = (\psi_\alpha)^\dagger$, where dagger denotes Hermitian conjugation of the field operator. If ψ_α is a classical Grassmann field, we use complex conjugation, i.e. $\bar\psi_{\dot\alpha} = (\psi_\alpha)^*$. The complex conjugation is defined in a such way that it has properties similar to the hermitian conjugation. For example, both of them reverse the order of fermions. The complex conjugate representation is

$$\bar\psi'_{\dot\alpha} = (M^*)_{\dot\alpha}{}^{\dot\beta} \bar\psi_{\dot\beta} \ . \tag{6}$$

$\bar\psi_{\dot\beta}$ is a right–handed Weyl spinor or dotted spinor. Left-handed spinor ψ_α is also called undotted spinor. These two representations of $SL(2,C)$ are inequivalent.

Now, we introduce invariant tensors for $SL(2,C)$. Antisymmetric tensors are defined by

$$\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = (\epsilon_{\alpha\beta})^{-1} = (\epsilon_{\dot\alpha\dot\beta})^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{8}$$

It is easy to prove $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$ and $\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \delta_{\dot\alpha}^{\dot\gamma}$. From

$$\epsilon_{\alpha\beta} M_\gamma{}^\alpha M_\delta{}^\beta = \epsilon_{\gamma\delta} \det M = \epsilon_{\gamma\delta} \tag{9}$$

it follows that $\epsilon_{\alpha\beta}$ is an invariant tensor. Similarly, $\epsilon^{\alpha\beta}, \epsilon_{\dot\alpha\dot\beta}$ and $\epsilon^{\dot\alpha\dot\beta}$ are invariant tensors. The invariant tensors are used for lowering or raising spinor indices:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \ , \tag{10}$$

$$\bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}, \quad \bar\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta} \ . \tag{11}$$

Let us find the transformation law for ψ^α:

$$\begin{aligned} \psi'^\alpha &= \epsilon^{\alpha\beta}\psi'_\beta = \epsilon^{\alpha\beta} M_\beta{}^\gamma \epsilon_{\gamma\delta}\psi^\delta \\ &= (\epsilon^{-1}M\epsilon)^\alpha{}_\delta \psi^\delta \\ &= (M^{T-1})^\alpha{}_\delta \psi^\delta \ . \end{aligned} \tag{12}$$

Spinors ψ_α and ψ^α transform according to equivalent representations. Similarly, we get

$$\bar\psi'^{\dot\alpha} = (M^{*-1})_{\dot\beta}{}^{\dot\alpha}\bar\psi^{\dot\beta} \ , \tag{13}$$

i. e. spinors $\bar\psi_{\dot\alpha}$ and $\bar\psi^{\dot\alpha}$ transform under equivalent representations. The fields $\bar\psi^{\dot\alpha}$ and $\bar\psi_{\dot\alpha}$ are right-handed Weyl spinors. Undotted and dotted representation are very often called fundamental and antifundamental representations. The left-handed (or undotted) spinors are $(\frac{1}{2}, 0)$ irreducible representation, while right-handed (or dotted) spinors are $(0, \frac{1}{2})$.

Multiplying these spinors with each other we get higher spinors. They have dotted and undotted indices, $\psi_{\alpha_1\ldots\alpha_n\dot\alpha_1\ldots\dot\alpha_m}$ and transform as follows

$$\psi'_{\alpha_1\ldots\alpha_n\dot\alpha_1\ldots\dot\alpha_m} = M_{\alpha_1}{}^{\beta_1}\ldots M_{\alpha_n}{}^{\beta_n}(M^*)_{\dot\alpha_1}{}^{\dot\beta_1}\ldots(M^*)_{\dot\alpha_m}{}^{\dot\beta_m}\psi_{\beta_1\ldots\beta_n\dot\beta_1\ldots\dot\beta_m} \ .$$

These spinors are generally reducible. The irreducible spinors are $\psi_{(\alpha_1,\ldots,\alpha_n)(\dot\alpha_1,\ldots,\dot\alpha_m)}$, where (..) denotes symmetrization. They belong to $\left(\frac{n}{2}, \frac{m}{2}\right)$ irreducible representation of $SL(2, C)$. Let us list some important irreducible representations:

- $(0, 0)$ scalar or singlet representation
- $(1/2, 0)$ left–handed Weyl spinor
- $(0, 1/2)$ right-handed Weyl spinor
- $(1/2, 1/2)$ vector
- $(1, 0)$ self-dual tensor
- $(0, 1)$ antiself-dual tensor.

From $M\sigma^\mu M^\dagger = (\Lambda^{-1})^\mu{}_\rho\sigma^\rho$ follows that matrices σ^μ are invariant tensors. The matrix σ^μ has the following spinorial indices

$$\sigma^\mu_{\alpha\dot\alpha} \ . \tag{14}$$

Raising the spinorial indices on σ^μ we obtain

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = \epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\beta\gamma}\sigma^\mu_{\gamma\dot{\gamma}} = (\sigma^\mu)^{\beta\dot{\alpha}} \ . \tag{15}$$

It is easy to show

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = (1, -\boldsymbol{\sigma})^{\dot{\alpha}\beta} \ . \tag{16}$$

The matrices σ^μ and $\bar{\sigma}^\mu$ satisfy the following identites:

$$\mathrm{Tr}(\sigma^\mu\bar{\sigma}^\nu) = 2g^{\mu\nu} \tag{17}$$

$$\sigma^\mu_{\alpha\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\beta}\beta} = \delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}} \tag{18}$$

$$\sigma^\mu_{\alpha\dot{\alpha}}(\bar{\sigma}_\mu)_{\dot{\beta}\beta} = -2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta} \tag{19}$$

$$(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_\alpha^{\ \beta} = 2g^{\mu\nu}\delta^\beta_\alpha \ . \tag{20}$$

Transformation laws for quantum Weyl fields are:

$$\begin{aligned}
U^{-1}(\Lambda)\psi_\alpha(x)U(\Lambda) &= M_\alpha^{\ \beta}\psi_\beta(\Lambda^{-1}x), \quad (\frac{1}{2}, 0) \\
U^{-1}(\Lambda)\psi^\alpha(x)U(\Lambda) &= (M^{-1})_\beta^{\ \alpha}\psi^\beta(\Lambda^{-1}x), \quad (\frac{1}{2}, 0) \\
U^{-1}(\Lambda)\bar{\psi}_{\dot{\alpha}}(x)U(\Lambda) &= (M^*)_{\dot{\alpha}}^{\ \dot{\beta}}\bar{\psi}_{\dot{\beta}}(\Lambda^{-1}x), \quad (0, \frac{1}{2}) \\
U^{-1}(\Lambda)\bar{\psi}^{\dot{\alpha}}(x)U(\Lambda) &= (M^{*-1})_{\dot{\beta}}^{\ \dot{\alpha}}\bar{\psi}^{\dot{\beta}}(\Lambda^{-1}x), \quad (0, \frac{1}{2}) \ ,
\end{aligned} \tag{21}$$

where $M \in \mathrm{SL}(2, \mathrm{C})$.

1.3 Dirac and Majorana spinors

Weyl representation of γ^μ matrices is given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \ . \tag{22}$$

γ_5 matrix is defined by

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{23}$$

Dirac spinor is

$$\Psi_D = \begin{pmatrix} \varphi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \ , \tag{24}$$

and it belong to the reducible representation $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$.

Adjoint Dirac spinor is

$$\bar{\psi}_D = \psi^\dagger\beta = \begin{pmatrix} \chi^\alpha & \bar{\varphi}_{\dot{\alpha}} \end{pmatrix} \tag{25}$$

where

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \delta^{\dot{\beta}}_{\dot{\alpha}} \\ \delta^{\beta}_{\alpha} & 0 \end{pmatrix} \ . \tag{26}$$

The matrices γ^0 and β are numerically equal, but have different spinorial structure.

Majorana spinor satisfies the condition $\Psi_M = C\bar{\Psi}^T_M$, were $C = i\gamma^0\gamma^2$ is the charge conjugation matrix. It is easy to show that

$$C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \ .$$

Majorana condition leads to $\psi_\alpha = \chi_\alpha$, so Majorana spinor is

$$\Psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \ . \tag{27}$$

1.4 Generators in (anti)fundamental representations

Under Lorentz transformation Dirac spinors transform as

$$\psi'_D(\Lambda x) = e^{-\frac{i}{4}\omega_{\mu\nu}\Sigma^{\mu\nu}}\psi_D(x) \ . \tag{28}$$

The matrices $\Sigma^{\mu\nu}$ are

$$\frac{1}{2}\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix} \ , \tag{29}$$

where

$$\begin{aligned} \sigma^{\mu\nu} &= \tfrac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu) \\ \bar{\sigma}^{\mu\nu} &= \tfrac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu) \ . \end{aligned}$$

Spinorial indices of these matrices are $(\sigma^{\mu\nu})_\alpha{}^\beta$ and $(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}$. Therefore, the M matrices are given by

$$\begin{aligned} M_\alpha{}^\beta &= \left(e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}\right)_\alpha{}^\beta \ , \\ (M^*)_{\dot{\alpha}}{}^{\dot{\beta}} &= \left(e^{\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}}\right)^{\dot{\beta}}{}_{\dot{\alpha}} \ , \\ (M^{-1T})^\alpha{}_\beta &= \left(e^{\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}\right)^\alpha{}_\beta \\ (M^{*-1})^{\dot{\alpha}}{}_{\dot{\beta}} &= \left(e^{-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}}\right)^{\dot{\alpha}}{}_{\dot{\beta}} \ . \end{aligned} \tag{30}$$

Matrices $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ are generators of Lorentz group, and they satisfy[1]

$$\begin{aligned} \epsilon_{\mu\nu\rho\sigma}\sigma^{\rho\sigma} &= 2i\sigma_{\mu\nu} \\ \epsilon_{\mu\nu\rho\sigma}\bar{\sigma}^{\rho\sigma} &= -2i\bar{\sigma}_{\mu\nu} \ . \end{aligned} \tag{31}$$

[1] We use the convention in which $\epsilon^{0123} = -\epsilon_{0123} = +1$

$\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ are a self-dual tensor and an anti-self-dual tensor, respectively. We can define $(\sigma^{\mu\nu}\epsilon)_{\alpha\beta}$ and $(\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}$ by

$$(\sigma^{\mu\nu}\epsilon)_{\alpha\beta} = (\sigma^{\mu\nu})_{\alpha}{}^{\gamma}\epsilon_{\gamma\beta} \ , \tag{32}$$

$$(\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\gamma}}(\bar{\sigma}^{\mu\nu})^{\dot{\gamma}}{}_{\dot{\beta}} \ . \tag{33}$$

They satisfy

$$\begin{aligned}(\sigma^{\mu\nu}\epsilon)_{\alpha\beta} &= (\sigma^{\mu\nu}\epsilon)_{\beta\alpha}\\ (\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} &= (\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\beta}\dot{\alpha}} \ .\end{aligned} \tag{34}$$

1.5 Bilinear quantities

Weyl spinors anticommute, i.e

$$\psi_\alpha\chi_\beta = -\chi_\beta\psi_\alpha \ . \tag{35}$$

Let us define the product of Weyl spinors in the following way

$$\psi\chi = \psi^\alpha\chi_\alpha = \psi^1\chi_1 + \psi^2\chi_2 = \psi_2\chi_1 - \psi_1\chi_2 \ . \tag{36}$$

Similarly, we introduce

$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \ . \tag{37}$$

In our convection dotted indices are contracted from lower left to upper right, while undotted indices are contracted from upper left to lower right. $\psi\chi$ and $\bar{\psi}\bar{\chi}$ are invariant under $SL(2,C)$. Four-vectors and tensors are defined as

$$\begin{aligned}\psi\sigma^\mu\bar{\chi} &= \psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \ ,\\ \bar{\psi}\bar{\sigma}^\mu\chi &= \bar{\psi}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\chi_\alpha \ ,\\ \psi\sigma^{\mu\nu}\chi &= \psi^\alpha(\sigma^{\mu\nu})_\alpha{}^\beta\chi_\beta \ ,\\ \bar{\psi}\bar{\sigma}^{\mu\nu}\bar{\chi} &= \bar{\psi}_{\dot{\alpha}}(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\chi}^{\dot{\beta}} \ .\end{aligned} \tag{38}$$

The bilinear quantities defined above satisfy the following properties:

$$\begin{aligned}\varphi\chi &= \chi\varphi \ ,\\ \bar{\varphi}\bar{\chi} &= \bar{\chi}\bar{\varphi} \ ,\\ \varphi\sigma^\mu\bar{\chi} &= -\bar{\chi}\bar{\sigma}^\mu\varphi \ ,\\ \varphi\sigma^\mu\bar{\sigma}^\nu\chi &= \chi\sigma^\nu\bar{\sigma}^\mu\varphi \ ,\\ \bar{\varphi}\bar{\sigma}^\mu\sigma^\nu\bar{\chi} &= \bar{\chi}\bar{\sigma}^\nu\sigma^\mu\bar{\varphi} \ ,\\ \varphi\sigma^{\mu\nu}\chi &= -\chi\sigma^{\mu\nu}\varphi \ ,\\ \bar{\varphi}\bar{\sigma}^{\mu\nu}\bar{\chi} &= -\bar{\chi}\bar{\sigma}^{\mu\nu}\bar{\varphi} \ ,\\ \varphi\sigma^{\mu\nu}\varphi &= 0 \ ,\\ \bar{\varphi}\bar{\sigma}^{\mu\nu}\bar{\varphi} &= 0 \ .\end{aligned} \tag{39}$$

Pauli matrices are hermitian. Their complex conjugation gives

$$(\sigma^{\mu}_{\alpha\dot{\beta}})^* = \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\beta\alpha}(\bar{\sigma}^{\mu})^{\dot{\beta}\alpha} = \bar{\sigma}^{\mu}_{\dot{\alpha}\beta} = \sigma^{\mu}_{\beta\dot{\alpha}} \ , \tag{40}$$

$$(\bar{\sigma}^{\mu\dot{\alpha}\beta})^* = \epsilon^{\alpha\gamma}\epsilon_{\dot{\beta}\dot{\delta}}(\sigma^{\mu})_{\gamma\dot{\delta}} = \sigma^{\mu\alpha\dot{\beta}} = \bar{\sigma}^{\mu\dot{\beta}\alpha} \ . \tag{41}$$

Taking complex conjugation of Lorentz generators we obtain

$$\begin{aligned} ((\sigma^{\mu\nu})_{\alpha}^{\ \beta})^* &= (\bar{\sigma}^{\mu\nu})^{\dot{\beta}}_{\ \dot{\alpha}} \ , \\ ((\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}})^* &= (\sigma^{\mu\nu})_{\beta}^{\ \alpha} \ . \end{aligned} \tag{42}$$

The following relations are useful:

$$((\sigma^{\mu\nu}\epsilon)_{\alpha\beta})^* = (\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\beta}\dot{\alpha}} \ , \tag{43}$$

$$((\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}})^* = (\sigma^{\mu\nu}\epsilon)_{\beta\alpha} \ . \tag{44}$$

Hermitian conjugation changes the order of spinors. For example:

$$(\psi\chi)^\dagger = (\psi^{\alpha}\chi_{\alpha})^\dagger = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = \bar{\psi}\bar{\chi} \ . \tag{45}$$

The following identities can be shown similarly:

$$\begin{aligned} (\bar{\psi}\bar{\chi})^\dagger &= \psi\chi \ , \\ (\psi\sigma^{\mu}\bar{\chi})^\dagger &= \chi\sigma^{\mu}\bar{\psi} \ , \\ (\bar{\psi}\bar{\sigma}^{\mu}\chi)^\dagger &= \bar{\chi}\bar{\sigma}^{\mu}\bar{\psi} \ , \\ (\psi\sigma^{\mu\nu}\chi)^\dagger &= -\bar{\psi}\bar{\sigma}^{\mu\nu}\bar{\chi} \ , \\ (\bar{\psi}\bar{\sigma}^{\mu\nu}\bar{\chi})^\dagger &= -\psi\sigma^{\mu\nu}\chi \ . \end{aligned} \tag{46}$$

1.6 Identites with σ matrices

The list of useful identities with σ matrices is given below:

$$\mathrm{Tr}(\sigma^{\mu}\bar{\sigma}^{\nu}) = 2g^{\mu\nu} \tag{47}$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}}(\bar{\sigma}_{\mu})^{\dot{\beta}\beta} = \delta^{\beta}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}} \tag{48}$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}}(\bar{\sigma}_{\mu})_{\dot{\beta}\beta} = -2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta} \tag{49}$$

$$(\sigma^{\mu}\bar{\sigma}^{\nu} + \sigma^{\nu}\bar{\sigma}^{\mu})_{\alpha}^{\ \beta} = 2g^{\mu\nu}\delta^{\beta}_{\alpha} \tag{50}$$

$$(\bar{\sigma}^{\mu}\sigma^{\nu} + \bar{\sigma}^{\nu}\sigma^{\mu})^{\dot{\alpha}}_{\ \dot{\beta}} = 2g^{\mu\nu}\delta^{\dot{\beta}}_{\dot{\alpha}} \tag{51}$$

$$(\sigma^{\mu}\bar{\sigma}^{\nu})_{\alpha}^{\ \beta} = g^{\mu\nu}\delta^{\beta}_{\alpha} - 2i(\sigma^{\mu\nu})_{\alpha}^{\ \beta} \tag{52}$$

$$(\bar{\sigma}^{\mu}\sigma^{\nu})^{\dot{\alpha}}_{\ \dot{\beta}} = g^{\mu\nu}\delta^{\dot{\beta}}_{\dot{\alpha}} - 2i(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \ . \tag{53}$$

From

$$\gamma^\mu\gamma^\nu\gamma^\rho = (g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\rho\nu})\gamma^\sigma + i\epsilon^{\sigma\mu\nu\rho}\gamma_5\gamma_\sigma \tag{54}$$

it follows

$$\begin{aligned}\sigma^\mu\bar\sigma^\nu\sigma^\rho &= g^{\mu\nu}\sigma^\rho - g^{\mu\rho}\sigma^\nu + g^{\nu\rho}\sigma^\mu - i\epsilon^{\sigma\mu\nu\rho}\sigma_\sigma \\ \bar\sigma^\mu\sigma^\nu\bar\sigma^\rho &= g^{\mu\nu}\bar\sigma^\rho - g^{\mu\rho}\bar\sigma^\nu + g^{\nu\rho}\bar\sigma^\mu + i\epsilon^{\sigma\mu\nu\rho}\bar\sigma_\sigma\ .\end{aligned} \tag{55}$$

The trace identities are:

$$\mathrm{Tr}(\sigma^{\mu\nu}\sigma^{\rho\sigma}) = \frac{1}{2}(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma} - i\epsilon^{\mu\nu\rho\sigma}) \tag{56}$$

$$\mathrm{Tr}(\bar\sigma^{\mu\nu}\bar\sigma^{\rho\sigma}) = \frac{1}{2}(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma} + i\epsilon^{\mu\nu\rho\sigma})\ . \tag{57}$$

1.7 Fierz identities

Fierz identities are very helpful in calculations with spinors. The following relations can be easily obtained

$$\begin{aligned}\theta^\alpha\theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta && (58)\\ \theta_\alpha\theta_\beta &= \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta \\ \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} &= \frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta \\ \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} &= -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta\ . && (59)\end{aligned}$$

With a help of the previous identities the following Fierz rearrangement formulas can be proven

$$\begin{aligned}(\theta\phi)(\theta\psi) &= -\frac{1}{2}(\theta\theta)(\phi\psi), && (60)\\ (\bar\theta\bar\phi)(\bar\theta\bar\psi) &= -\frac{1}{2}(\bar\theta\bar\theta)(\bar\phi\bar\psi), \\ (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) &= \frac{1}{2}g^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta), \\ (\theta\sigma^\mu\bar\theta)\theta_\alpha &= -\frac{1}{2}\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}(\theta\theta), \\ (\theta\sigma^\mu\bar\theta)\bar\theta_{\dot\alpha} &= \frac{1}{2}\sigma^\mu_{\beta\dot\alpha}\theta^\beta(\bar\theta\bar\theta), \\ (\bar\theta\bar\lambda)(\chi\sigma^\mu\bar\theta) &= -\frac{1}{2}(\bar\theta\bar\theta)(\chi\sigma^\mu\bar\lambda), \\ (\theta\lambda)(\bar\chi\bar\sigma^\mu\theta) &= -\frac{1}{2}(\theta\theta)(\bar\chi\bar\sigma^\mu\lambda), \\ (\theta\sigma^\nu\bar\psi)(\theta\sigma^\mu\bar\lambda) &= \frac{1}{2}(\theta\theta)(\bar\psi\bar\sigma^\nu\sigma^\mu\bar\lambda), && (61)\end{aligned}$$

$$\chi_\alpha(\xi\eta) + \xi_\alpha(\eta\chi) + \eta_\alpha(\chi\xi) = 0, \tag{62}$$

$$\bar\chi_{\dot\alpha}(\bar\xi\bar\eta) + \bar\xi_{\dot\alpha}(\bar\eta\bar\chi) + \bar\eta_{\dot\alpha}(\bar\chi\bar\xi) = 0 \ . \tag{63}$$

As a example, let us check (60). Applying (58), the left hand side of (60) becomes

$$(\theta\phi)(\theta\psi) = \frac{1}{2}\epsilon^{\alpha\beta}(\theta\theta)\phi_\alpha\psi_\beta = -\frac{1}{2}(\theta\theta)(\phi\psi) \ . \tag{64}$$

Remaining identities can be proven in a similar way.

Let us generalize identities (58-59) to the case of two different spinors. We obtain

$$\theta_\alpha\bar\psi_{\dot\alpha} = \frac{1}{2}(\theta\sigma^\mu\bar\psi)(\sigma_\mu)_{\alpha\dot\alpha} \tag{65}$$

$$\begin{aligned}
\psi_\alpha\chi_\beta &= \frac{1}{2}\epsilon_{\alpha\beta}\psi\chi - \frac{1}{2}(\psi\sigma^{\mu\nu}\chi)(\sigma_{\mu\nu}\epsilon)_{\alpha\beta} \\
\bar\psi_{\dot\alpha}\bar\chi_{\dot\beta} &= -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\psi\bar\chi - \frac{1}{2}(\bar\chi\bar\sigma^{\mu\nu}\bar\psi)(\epsilon\bar\sigma_{\mu\nu})_{\dot\beta\dot\alpha} \ .
\end{aligned} \tag{66}$$

Let us prove (65). The left hand side of (65) is a product of dotted and undotted spinors. Since

$$(\frac{1}{2},0)\otimes(0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$$

we can assume

$$\theta_\alpha\bar\psi_{\dot\alpha} = A^\mu\sigma_{\mu\alpha\dot\alpha} \ , \tag{67}$$

where A^μ is a four-vector. Multiplying (67) by $\bar\sigma^{\nu\dot\alpha\alpha}$ we obtain

$$A^\mu = \frac{1}{2}\theta\sigma^\mu\bar\psi \ .$$

1.8 Partial derivatives

Partial derivatives with respect to the anticommuting variables are denoted by

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}, \ \partial^\alpha = \frac{\partial}{\partial\theta_\alpha}, \ \bar\partial_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}}, \bar\partial^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} \ . \tag{68}$$

Indices on partial derivatives are lowered or raised by $\epsilon-$ symbol. Note that, there is one extra minus sign:

$$\begin{aligned}
\partial_\alpha = -\epsilon_{\alpha\beta}\partial^\beta, \partial^\alpha = -\epsilon^{\alpha\beta}\partial_\beta \\
\bar\partial_{\dot\alpha} = -\epsilon_{\dot\alpha\dot\beta}\bar\partial^{\dot\beta}, \bar\partial^{\dot\alpha} = -\epsilon^{\dot\alpha\dot\beta}\bar\partial_{\dot\beta} \ .
\end{aligned} \tag{69}$$

The following derivatives are frequently used:

$$\frac{\partial\theta^\beta}{\partial\theta^\alpha} = \partial_\alpha\theta^\beta = \delta^\beta_\alpha, \ \frac{\partial\theta_\beta}{\partial\theta^\alpha} = \partial_\alpha\theta_\beta = \epsilon_{\beta\alpha} \tag{70}$$

$$\bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta^{\dot\beta}_{\dot\alpha}, \ \bar\partial_{\dot\alpha}\bar\theta_{\dot\beta} = \epsilon_{\dot\beta\dot\alpha} \tag{71}$$

$$\partial_\alpha(\theta\theta) = 2\theta_\alpha \tag{72}$$
$$\bar{\partial}_{\dot{\alpha}}(\bar{\theta}\bar{\theta}) = -2\bar{\theta}_{\dot{\alpha}} \tag{73}$$
$$\partial^\alpha(\theta\theta) = -2\theta^\alpha \tag{74}$$
$$\bar{\partial}^{\dot{\alpha}}(\bar{\theta}\bar{\theta}) = 2\bar{\theta}^{\dot{\alpha}} \tag{75}$$
$$(\partial\partial)(\theta\theta) = \partial^\alpha\partial_\alpha(\theta\theta) = 4 \tag{76}$$
$$(\bar{\partial}\bar{\partial})(\bar{\theta}\bar{\theta}) = \bar{\partial}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}}(\bar{\theta}\bar{\theta}) = 4 \ . \tag{77}$$

Complex conjugation of derivative is defined by

$$(\partial_\alpha)^* = -\bar{\partial}_{\dot{\alpha}} \ . \tag{78}$$

This is in accordance with definition of conjugation

$$\delta_{\dot{\alpha}}^{\dot{\beta}} = (\partial_\alpha\theta^\beta)^* = \bar{\theta}^{\dot{\beta}}(\partial_\alpha)^* = -(\partial_\alpha)^*\bar{\theta}^{\dot{\beta}} = \bar{\partial}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} \ . \tag{79}$$

2 Supersymmetric algebra

2.1 $N = 1$ Super–Poincare algebra

To start with, let us state the Coleman-Mandula theorem:

Only possible Lie symmetry of a nontrivial S matrix is a direct product of Poincare and internal symmetry.

The assumptions in Coleman-Mandula theorem are the following: the particles are massive, theory is local, unitary and relativistic covariant. Thus, generators of the Lie symmetry are: momenta, P^μ, angular momenta, $M_{\mu\nu}$ and bosonic generators B^r. These bosonic generators are Lorentz scalars.

This theorem can be evaded using the graded Lie symmetry instead of the usual Lie symmetry. The graded Lie algebra is a generalization of Lie algebra, containing bosonic and fermionic generators. The graded vector space (precisely Z_2 graded) is a direct sum

$$V = L_0 \oplus L_1 \ .$$

The elements of L_0 are bosons or even elements, while the elements of L_1 are fermions or odd elements. The grading is defined by

$$|X| = \begin{cases} 0, & X \in L_0 \\ 1, & X \in L_1 \end{cases} \ . \tag{1}$$

Between elements of the graded vector space Lie product $[,\}$ is defined, with the following properties:

$$\begin{aligned} &1.\ [X,Y\} = -(-)^{|X||Y|}[Y,X\} \ , \\ &2.\ (-)^{|X||Z|}[X,[Y,Z\}\} + (-)^{|Y||X|}[Y,[Z,X\}\} + (-)^{|Z||Y|}[Z,[X,Y\}\} = 0 \ . \end{aligned} \tag{2}$$

The Lie product of two bosons or one fermion and one boson is a commutator. In the case two fermions Lie product is an anticommutator, i.e.

$$[B_1, B_2\} = [B_1, B_2],\ [B_1, F_1\} = [B_1, F_1],\ [F_1, F_2\} = \{F_1, F_2\}\ . \tag{3}$$

The second property is super-Jacobi identity.

Super–Poincare algebra is an extension of Poincare algebra which includes the fermionic generators in addition to the bosonic ones. These fermionic generators transform bosonic state into fermionic and vice verse. The fermionic generators are left-handed and right-handed spinors, Q_α and $\bar{Q}_{\dot\alpha}$. This statement is known as the Haag–Loppusanski–Sonius theorem.

The Lie product of generators is a commutator or anticommutator, and it must be linear in terms of generators. First, we have to find $[P_\mu, Q_\alpha]$. We assume that

$$[P^\mu, Q_\alpha] = c\sigma^\mu_{\alpha\dot\beta}\bar{Q}^{\dot\beta}\ . \tag{4}$$

From this expression it follows that

$$[P^\mu, \bar{Q}^{\dot\beta}] = c^*\bar\sigma^{\mu\dot\beta\gamma}Q_\gamma\ . \tag{5}$$

Applying super–Jacobi identity

$$[P^\mu, [P^\nu, Q_\alpha]] + [P^\nu, [Q_\alpha, P^\mu]] + [Q_\alpha, [P^\mu, P^\nu]] = 0$$

we obtain

$$|c|^2(\sigma^{\nu\mu})_\alpha^{\ \beta}Q_\beta = 0\ ,$$

and finally we conclude $c = 0$, i. e. $[P_\mu, Q_\alpha] = 0$.

Since Q_α is a Weyl spinor, it transforms as follows

$$e^{\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}}Q_\alpha e^{-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}} = \left(e^{-i\frac{1}{2}\omega^{\mu\nu}\sigma_{\mu\nu}}\right)_\alpha^{\ \beta}Q_\beta\ .$$

Expanding the previous expression in the first order in $\omega_{\mu\nu}$ we obtain

$$[M_{\mu\nu}, Q_\alpha] = -(\sigma_{\mu\nu})_\alpha^{\ \beta}Q_\beta\ . \tag{6}$$

In the same way, one can show that

$$[M_{\mu\nu}, \bar{Q}^{\dot\alpha}] = -(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\ \dot\beta}\bar{Q}^{\dot\beta}\ . \tag{7}$$

Let us find the anticommutator $\{Q_\alpha, Q_\beta\}$. Due to index structure this anticommutator has the form

$$\{Q_\alpha, Q^\beta\} = d(\sigma^{\mu\nu})_\alpha^{\ \beta}M_{\mu\nu}\ , \tag{8}$$

where d is an arbitrary constant. Applying the super–Jacobi identity

$$[P_\mu, \{Q_a, Q^\beta\}] + \{Q_\alpha, [Q^\beta, P_\mu]\} - \{Q^\beta, [P_\mu, Q_\alpha]\} = 0$$

we find $d = 0$. Thus, $\{Q_\alpha, Q^\beta\} = 0$. Finally, we have to find $\{Q_\alpha, \bar{Q}_{\dot\beta}\}$. Taking into account the index structure of this expression we find

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu{}_{\alpha\dot\beta} P_\mu \ . \tag{9}$$

The factor 2 is fixed by a convection.

Simple (or $N = 1$) Super–Poincare algebra is

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, && (10)\\
[M_{\mu\nu}, P_\rho] &= i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu), && (11)\\
[M_{\mu\nu}, M_{\rho\sigma}] &= -i(g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma} - g_{\mu\sigma}M_{\nu\rho}), && (12)\\
\{Q_\alpha, Q_\beta\} &= \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0, &&\\
\{Q_\alpha, \bar{Q}_{\dot\beta}\} &= 2\sigma^\mu{}_{\alpha\dot\beta} P_\mu, && (13)\\
[Q_\alpha, P_\mu] &= [\bar{Q}_{\dot\alpha}, P_\mu] = 0, &&\\
[Q_\alpha, M_{\mu\nu}] &= (\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta, &&\\
[\bar{Q}^{\dot\alpha}, M_{\mu\nu}] &= (\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \bar{Q}^{\dot\beta}. &&
\end{aligned}$$

2.2 Extended Super–Poincare algebra

Fermionic generators can have an additional index coming from some internal symmetry group: Q^A_α $(A = 1, \ldots, \mathcal{N})$. In this case $N = 1$ supersymmetry is extended to the $\mathcal{N}-$extended supersymmetry:

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, && (14)\\
[M_{\mu\nu}, P_\rho] &= i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu), &&\\
[M_{\mu\nu}, M_{\rho\sigma}] &= -i(g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma} - g_{\mu\sigma}M_{\nu\rho}), &&\\
\{Q^A_\alpha, Q^B_\beta\} &= \epsilon_{\alpha\beta} Z^{AB} \ , &&\\
\{\bar{Q}^A_{\dot\alpha}, \bar{Q}^B_{\dot\beta}\} &= \epsilon_{\dot\alpha\dot\beta} Z^{\dagger AB}, &&\\
\{Q^A_\alpha, \bar{Q}^B_{\dot\beta}\} &= 2\delta^{AB}\sigma^\mu{}_{\alpha\dot\beta} P_\mu, &&\\
[Q^A_\alpha, P_\mu] &= [\bar{Q}^A_{\dot\alpha}, P_\mu] = 0, &&\\
[Q^A_\alpha, M_{\mu\nu}] &= (\sigma_{\mu\nu})_\alpha{}^\beta Q^A_\beta, &&\\
[\bar{Q}^{\dot\alpha A}, M_{\mu\nu}] &= (\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \bar{Q}^{\dot\beta A} &&\\
[B^r, B^s] &= i f^{rst} B^t &&\\
[B^r, Q^A_\alpha] &= -(b^r)^A{}_C Q^C_\alpha &&\\
[B^r, \bar{Q}^A_{\dot\alpha}] &= \bar{Q}_{\dot\alpha C}(b^r)^C{}_A &&
\end{aligned}$$

$$\tag{15}$$

Central charge, $Z^{AB} = -Z^{BA}$ commutes with all generators in the algebra. It is a linear combination of the internal symmetry generators, $Z^{AB} = \lambda^{ABr} B_r$. The largest possible internal symmetry group is $U(N)$. If $Z=0$, SUSY algebra is invariant under $Q^A_\alpha \longrightarrow U^A{}_B\, Q^B_\alpha$. If $Z \neq 0$, $\{Q,Q\}$ relation spoils this invariance. In this case the algebra is invariant under some subgroup of $U(N)$. In $N=1$ SUSY, central charge is zero, $Z=0$: there is only one index, so Z can't be antisymmetric. In this case, $U(1)$ symmetry is called *R-symmetry*:

$$[Q_\alpha, R] = Q_\alpha, \qquad [\bar{Q}_{\dot\alpha}, R] = -\bar{Q}_{\dot\alpha}. \tag{16}$$

3 Superspace and Superfields

In this section we introduce the concept of superspace. Fields on superspace are called superfields. Supersymmetry is realized as a translation in superspace.

A superspace is an extension of Minkowski space-time where we add four Grassmann coordinates $\theta_\alpha, \bar\theta_{\dot\alpha}$ to the four bosonic Minkowski coordinates. These coordinates satisfy

$$\begin{aligned}&[x^\mu, x^\nu] = [x_\mu, \theta_\alpha] = [x^\mu, \bar\theta_{\dot\alpha}] = 0\\ &\{\theta_\alpha, \bar\theta_{\dot\alpha}\} = \{\theta_\alpha, \theta_\beta\} = \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = 0\ .\end{aligned} \tag{1}$$

Thus, superspace is parametrized by the coordinates: $x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha}$. For later convenience we calculate

$$\begin{aligned}[\xi\mathbb{Q}, \bar\theta\bar{\mathbb{Q}}] \quad &= \xi^\alpha \mathbb{Q}_\alpha \bar\theta_{\dot\alpha}\bar{\mathbb{Q}}^{\dot\alpha} - \bar\theta_{\dot\alpha}\bar{\mathbb{Q}}^{\dot\alpha}\xi^\alpha\mathbb{Q}_\alpha\\ &= -\xi^\alpha\bar\theta_{\dot\alpha}\mathbb{Q}_\alpha\bar{\mathbb{Q}}^{\dot\alpha} + \bar\theta_{\dot\alpha}\xi^\alpha\bar{\mathbb{Q}}^{\dot\alpha}\mathbb{Q}_\alpha\\ &= -\xi^\alpha\bar\theta_{\dot\alpha}\{\mathbb{Q}_\alpha, \bar{\mathbb{Q}}^{\dot\alpha}\}\\ &= \xi^\alpha\bar\theta^{\dot\alpha}\{\mathbb{Q}_\alpha, \bar{\mathbb{Q}}_{\dot\alpha}\}\\ &= 2\xi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}P_\mu\ .\end{aligned} \tag{2}$$

In this part of Lectures, SUSY generators are denoted by $\mathbb{Q}, \bar{\mathbb{Q}}$. Any element of super–Poincare group can be written in the form

$$\mathrm{e}^{i(x^\mu P_\mu + \theta^\alpha\mathbb{Q}_\alpha + \bar\theta_{\dot\alpha}\bar{\mathbb{Q}}^{\dot\alpha})}\mathrm{e}^{-\frac{1}{2}\omega^{\mu\nu}M_{\mu\nu}}\ . \tag{3}$$

We use the notation

$$G(x,\theta,\bar\theta) = \mathrm{e}^{i(x^\mu P_\mu + \theta^\alpha\mathbb{Q}_\alpha + \bar\theta_{\dot\alpha}\bar{\mathbb{Q}}^{\dot\alpha})}\ . \tag{4}$$

Actually, $G(x,\theta,\bar\theta)$ is an element of the coset space

$$\text{Super} - \text{Poincare group/Lorentz group}\ .$$

A superspace can be realized as a coset space. For details, see [1, 5]. The product of two elements, $G(a,\xi,\bar\xi)G(x,\theta,\bar\theta)$ is found applying Baker–Campbell–Hausdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}$$

and (2). The result is

$$G(a,\xi,\bar\xi)G(x,\theta,\bar\theta) = G(x^\mu + a^\mu + i\xi\sigma^\mu\bar\theta - i\theta\sigma^\mu\bar\xi, \theta+\xi, \bar\theta+\bar\xi)\ . \tag{5}$$

The left multiplication in the group induces the following transformation of supercoordinates:

$$\begin{aligned} x^\mu &\to x^\mu + \delta x^\mu = x^\mu + a^\mu + i\xi\sigma^\mu\bar\theta - i\theta\sigma^\mu\bar\xi \\ \theta &\to \theta + \xi \\ \bar\theta &\to \bar\theta + \bar\xi \ , \end{aligned}$$

where ξ is a constant parameter of SUSY transformation.

Under infinitesimal translation $x' = x + a$ a classical scalar field $\varphi(x)$ transforms as $\varphi(x) \to \varphi'(x) = \varphi(x - a)$. The form-variation of the scalar field is given by

$$\delta\varphi(x) = \varphi'(x) - \varphi(x) = -a^\mu \partial_\mu \varphi(x) \ . \tag{6}$$

From $\delta\varphi(x) = ia^\mu P_\mu \varphi$ we conclude that $P^\mu = i\partial^\mu$ is a representation of momenta in the space of functions. The momentum is represented as a differential operator in accordance to quantum mechanics.

In QFT a scalar field is an operator in Hilbert space. Transformation law under translations is given by

$$\varphi(x) \to \varphi'(x) = e^{-ia^\mu P_\mu}\varphi(x) e^{ia^\mu P_\mu} = \varphi(x - a) \ , \tag{7}$$

where P^μ are four-momenta. Then we obtain

$$\delta\varphi = \varphi'(x) - \varphi(x) = -i[a^\mu P_\mu, \varphi] = -a^\mu \partial_\mu \varphi \ . \tag{8}$$

Generally, infinitesimal variation of field under transformations of symmetry is given by

$$\delta\varphi = i[\int d^3x j^0, \varphi] \ , \tag{9}$$

where j^0 is zero component of the Noether current density. Variations of a quantum and classical field are related by the correspondence principle. If the canonical Poisson bracket of two observable gives a third observable

$$\{A, B\} = C \ ,$$

then after quantization the commutator of corresponding operators has to satisfy

$$-i[\hat A, \hat B] = \hat C \ .$$

For details see the book [6].

A 'scalar' superfield is a function of superspace coordinates, $F = F(x, \theta, \bar\theta)$. A superfield can be expanded in power series of θ and $\bar\theta$ as

$$\begin{aligned} F(x,\theta,\bar\theta) &= f(x) + \theta\phi(x) + \bar\theta\bar\chi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x) + \theta\sigma^\mu\bar\theta v_\mu(x) \\ &\quad + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\varphi(x) + \theta\theta\bar\theta\bar\theta d(x). \end{aligned} \tag{10}$$

This expansion consists of finite number of terms, due to the anticommuting properties of $\theta, \bar\theta$ coordinates. The coefficients $f(x)$, $\phi(x)$, $\bar\chi(x)$, $m(x)$, $n(x)$, $v_\mu(x)$, $\bar\lambda(x)$, $\varphi(x)$ and $d(x)$ in the expansion (10) are usual fields in Minkowski space–time. Superfield (10) contains 16 fermionic and 16 bosonic degrees of freedom. There is an equal number of bosonic and fermionic degrees of freedom in any supersymmetric multiplet.

A superfield has to satisfy some transformation properties. Under super–Poincare transformation[2]

$$\begin{aligned} x^\mu &\rightarrow x' = x^\mu - \delta x^\mu = x^\mu - a^\mu - i\xi\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\xi} \\ \theta &\rightarrow \theta' = \theta - \xi \\ \bar{\theta} &\rightarrow \bar{\theta}' = \bar{\theta} - \bar{\xi} \ , \end{aligned} \tag{11}$$

a classical scalar superfield transforms as

$$F'(x,\theta,\bar{\theta}) = F(x+\delta x, \theta+\xi, \bar{\theta}+\bar{\xi}) \ . \tag{12}$$

The infinitesimal variation of the superfield is given by

$$\begin{aligned} \delta F &= F'(x,\theta,\bar{\theta}) - F(x,\theta,\bar{\theta}) \\ &= F(x + a^\mu + i\xi\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\xi}, \theta+\xi, \bar{\theta}+\bar{\xi}) - F(x,\theta,\bar{\theta}) \\ &= (a^\mu + i\xi\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\xi})\partial_\mu F + \xi^\alpha\partial_\alpha F + \bar{\xi}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}} F \\ &= (a^\mu\partial_\mu + \xi^\alpha Q_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}})F \ , \end{aligned} \tag{13}$$

where we introduce the differential operators:

$$\begin{aligned} Q_\alpha &= \partial_\alpha + i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu \\ \bar{Q}^{\dot{\alpha}} &= \bar{\partial}^{\dot{\alpha}} + i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\theta_\alpha\partial_\mu \ . \end{aligned} \tag{14}$$

From[3] $\delta F = -i(a^\mu P_\mu + \xi^\alpha \mathbb{Q}_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{\mathbb{Q}}^{\dot{\alpha}})F$ it follows

$$P_\mu = i\partial_\mu, \tag{15}$$

$$\mathbb{Q}_\alpha = iQ_\alpha = i\partial_\alpha - \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \ , \tag{16}$$

$$\bar{\mathbb{Q}}^{\dot{\alpha}} = i\bar{Q}^{\dot{\alpha}} = i\bar{\partial}^{\dot{\alpha}} - (\bar{\sigma}^\mu\theta)^{\dot{\alpha}}\partial_\mu. \tag{17}$$

One can check that

$$\bar{\mathbb{Q}}_{\dot{\alpha}} = (\mathbb{Q}_\alpha)^\dagger = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\mathbb{Q}}^{\dot{\beta}} = i\bar{Q}_{\dot{\alpha}} \ ,$$

where $Q_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - (\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu$. The operators $P_\mu, \mathbb{Q}_\alpha$ and $\bar{\mathbb{Q}}_{\dot{\alpha}}$ satisfy super–Poincare algebra, as we expect. For example,

$$\{\mathbb{Q}_\alpha, \bar{\mathbb{Q}}_{\dot{\alpha}}\} = -\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} i\partial_\mu = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \ . \tag{18}$$

Under supertanslation (11) the quantum superfield $F(x,\theta,\bar{\theta})$ transforms as

$$\begin{aligned} & \mathrm{e}^{i(a^\mu P_\mu + \xi^\alpha\mathbb{Q}_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{\mathbb{Q}}^{\dot{\alpha}})} F(x,\theta,\bar{\theta}) \mathrm{e}^{-i(a^\mu P_\mu + \xi^\alpha\mathbb{Q}_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{\mathbb{Q}}^{\dot{\alpha}})} \\ &= F(x + a^\mu + i\xi\sigma^\mu\bar{\theta} - i\theta\sigma^\mu\bar{\xi}, \theta+\xi, \bar{\theta}+\bar{\xi}) \\ &= F + \delta F \ . \end{aligned} \tag{19}$$

[2] Note minus signs in coordinate transformations.

[3] The parameters are $-a, -\xi, -\bar{\xi}$.

Expanding the first line of (19) we obtain

$$
\begin{aligned}
&\mathrm{e}^{i(a^\mu P_\mu+\xi^\alpha \mathbb{Q}_\alpha+\bar{\xi}_{\dot\alpha}\bar{\mathbb{Q}}^{\dot\alpha})}F(x,\theta,\bar\theta)\mathrm{e}^{-i(a^\mu P_\mu+\xi^\alpha \mathbb{Q}_\alpha+\bar{\xi}_{\dot\alpha}\bar{\mathbb{Q}}^{\dot\alpha})} = \\
&F(x,\theta,\bar\theta)+i[a^\mu P_\mu+\xi^\alpha\mathbb{Q}_\alpha+\bar\xi_{\dot\alpha}\bar{\mathbb{Q}}^{\dot\alpha},F]\ .
\end{aligned} \tag{20}
$$

From the second line in (19) we again obtain

$$
\delta F=(a^\mu\partial_\mu+\xi^\alpha Q_\alpha+\bar\xi_{\dot\alpha}\bar Q^{\dot\alpha})F\ .
$$

Under the SUSY infinitesimal transformation the variation of of quantum superfields is given by

$$
\delta_\xi F=i[\xi^\alpha\mathbb{Q}_\alpha+\bar\xi_{\dot\alpha}\bar{\mathbb{Q}}^{\dot\alpha},F]=(\xi^\alpha Q_\alpha+\bar\xi_{\dot\alpha}\bar Q^{\dot\alpha})F\ . \tag{21}
$$

Applying (10) we find SUSY variations for components in supermultiplet:

$$
\begin{aligned}
\delta_\xi f &= \xi\phi+\bar\xi\bar\chi, &(22)\\
\delta_\xi\phi_\alpha &= 2\xi_\alpha m+\sigma^\mu{}_{\alpha\dot\alpha}\bar\xi^{\dot\alpha}\Big(v_\mu-i\partial_\mu f\Big), &(23)\\
\delta_\xi\bar\chi_{\dot\alpha} &= 2\bar\xi_{\dot\alpha}n+\xi^\alpha\sigma^\mu_{\alpha\dot\alpha}\big(v_\mu+i\partial_\mu f\big), &(24)\\
\delta_\xi m &= \bar\xi\bar\lambda-\frac{i}{2}\bar\xi\bar\sigma^\mu\partial_\mu\phi, &(25)\\
\delta_\xi n &= \xi\varphi-\frac{i}{2}\xi\sigma^\mu\partial_\mu\bar\chi, &(26)\\
\delta_\xi v_\mu &= \xi\sigma_\mu\bar\lambda+\varphi\sigma_\mu\bar\xi-\frac{i}{2}\xi\sigma_\nu\bar\sigma_\mu\partial^\nu\phi+\frac{i}{2}\bar\xi\bar\sigma_\nu\sigma_\mu\partial^\nu\bar\chi, &(27)\\
\delta_\xi\bar\lambda^{\dot\alpha} &= 2\bar\xi^{\dot\alpha}d-i(\bar\sigma^\mu\xi)^{\dot\alpha}(\partial_\mu m)-\frac{i}{2}(\bar\sigma^\nu\sigma^\mu\bar\xi)^{\dot\alpha}\partial_\mu v_\nu, &(28)\\
\delta_\xi\varphi_\alpha &= 2\xi_\alpha d-i\sigma^\mu{}_{\alpha\dot\alpha}\bar\xi^{\dot\alpha}\partial_\mu n+\frac{i}{2}\sigma^\nu{}_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\xi_\beta\partial_\mu v_\nu, &(29)\\
\delta_\xi d &= -\frac{i}{2}\xi\sigma^\mu\partial_\mu\bar\lambda+\frac{i}{2}\partial_\mu\varphi\sigma^\mu\bar\xi. &(30)
\end{aligned}
$$

Note that a SUSY variation of the highest component, $d(x)$ is a total derivative.

Let us apply two successive supersymmetric transformations on a superfield, $\delta_\eta\delta_\xi F$. Transformation with the parameter ξ, is followed by the second transformation with parameter η. The second transformation acts only on the field F, i.e.

$$
\delta_\eta\delta_\xi F=\delta_\eta(\xi Q+\bar\xi\bar Q)F=(\xi Q+\bar\xi\bar Q)(\eta Q+\bar\eta\bar Q)F\ .
$$

A commutator of two SUSY transformations can be found easily. Result is given by

$$
[\delta_\eta,\delta_\xi]F=-2i(\xi\sigma^\mu\bar\eta-\eta\sigma^\mu\bar\xi)\partial_\mu F\ . \tag{31}
$$

We see that commutator of two SUSY transformations is a translation, i.e. the supersymmetric algebra is closed.

3.1 Covariant Fermionic Derivatives

Supercovariant derivatives are defined by

$$D_\alpha = \partial_\alpha - i\sigma^\mu{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu, \tag{32}$$

$$\bar D_{\dot\alpha} = -\bar\partial_{\dot\alpha} + i\theta^\alpha\sigma^\mu{}_{\alpha\dot\alpha}\partial_\mu. \tag{33}$$

With a help of $(\partial_\alpha)^* = -\bar\partial_{\dot\alpha}$ and $(\sigma^\mu_{\alpha\dot\beta})^*\theta^\beta = \sigma^\mu_{\beta\dot\alpha}\theta^\beta$ we get $(D_\alpha)^* = \bar D_{\dot\alpha}$. Derivatives with upper indices are

$$\begin{aligned} D^\alpha &= \epsilon^{\alpha\beta}D_\beta = -\partial^\alpha + i\bar\theta_{\dot\beta}\bar\sigma^{\mu\dot\beta\alpha}\partial_\mu \\ \bar D^{\dot\alpha} &= \bar\partial^{\dot\alpha} - i\sigma^{\mu\dot\alpha\alpha}\theta_\alpha\partial_\mu \ . \end{aligned} \tag{34}$$

The following anticommutation relations are very useful:

$$\{D_\alpha, D_\beta\} = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = \{D_\alpha, Q_\beta\} = \{D_\alpha, \bar Q_{\dot\alpha}\} = \{\bar D_{\dot\alpha}, Q_\alpha\} = \{\bar D_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0, \tag{35}$$

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = -\{D_\alpha, \bar D_{\dot\alpha}\} = -2i\sigma^\mu{}_{\alpha\dot\alpha}\partial_\mu. \tag{36}$$

One can check that:

$$D^2 = D^\alpha D_\alpha = -\partial^\alpha\partial_\alpha - 2i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\partial^\alpha + \bar\theta\bar\theta\Box \ , \tag{37}$$

$$\bar D^2 = \bar D_{\dot\alpha}\bar D^{\dot\alpha} = -\bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha} - 2i\bar\sigma^{\mu\dot\alpha\alpha}\theta_\alpha\partial_\mu\bar\partial_{\dot\alpha} + \theta\theta\Box \ . \tag{38}$$

3.2 Chiral and antichiral superfields

An arbitrary superfield (10) is a reducible supersymmetric multiplet. One can impose some constraints on the superfield to get irreducible representations of supersymmetry. A chiral superfield $\Phi(x,\theta,\bar\theta)$ is defined by the following constraint $\bar D_{\dot\alpha}\Phi(x,\theta,\bar\theta) = 0$, and it is a irreducible superfield. This constraint is compatible with supersymmetry,

$$\delta_\xi(\bar D_{\dot\alpha}\Phi) = \bar D_{\dot\alpha}(\delta_\xi\Phi), \tag{39}$$

since supercovariant derivative anticommutes with supercharges.

We now want to solve this constraint. It can be proven that

$$\begin{aligned} \bar D_{\dot\alpha}F &= \bar\chi_{\dot\alpha} + \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}(v^\mu + i\partial_\mu f) + 2n\bar\theta_{\dot\alpha} + (\theta\theta)\left(\bar\lambda_{\dot\alpha} - \frac{i}{2}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\phi^\alpha\right) \\ &+ (\theta\sigma^\mu\bar\theta)\left(\sigma_{\mu\beta\dot\alpha}\varphi^\beta - \frac{i}{2}(\partial_\nu\bar\chi\bar\sigma_\mu\sigma^\nu)_{\dot\alpha}\right) \\ &+ 2(\theta\theta)\bar\theta_{\dot\alpha}d - \frac{i}{2}\epsilon^{\alpha\beta}(\theta\theta)\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\bar\theta^{\dot\beta}\partial_\mu v_\nu \\ &+ i(\bar\theta\bar\theta)\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu n - \frac{i}{2}(\theta\theta)(\bar\theta\bar\theta)\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\varphi^\alpha \ . \end{aligned} \tag{40}$$

The condition $\bar D_{\dot\alpha}F = 0$ implies

$$\bar\chi = 0, \quad n = 0, \quad v_\mu = -i\partial_\mu f, \quad \varphi = 0, \quad \bar\lambda_{\dot\alpha} = \frac{i}{2}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\phi^\alpha, \quad d = -\frac{1}{4}\Box f \ . \tag{41}$$

Introducing new notation:

$$f \equiv A, \qquad \phi \equiv \sqrt{2}\psi, \qquad m \equiv F \tag{42}$$

a general expression for a chiral superfield is

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) &= A(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu A(x) \\ &+ \frac{i}{\sqrt{2}}(\theta\theta)(\partial_\mu\psi(x)\sigma^\mu\bar{\theta}) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\Box A(x)\ .\end{aligned} \tag{43}$$

The chiral supermultiplet consists of a complex scalar field A (2 d.o.f.), a left-handed Weyl-spinor ψ_α (4 d.o.f.) and a complex scalar field F (2 d.o.f). Notice that the number of fermionic and bosonic d.o.f. is equal.

By introducing chiral supercoordinate:

$$y^\mu = x^\mu - i(\theta\sigma^\mu\bar{\theta})\ , \tag{44}$$

we can rewrite the chiral superfield in the form:

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)\ . \tag{45}$$

From general transformation rules (22-30), one can easily find transformation laws for components of a chiral multiplet:

$$\begin{aligned}\delta_\xi A &= \sqrt{2}\xi\psi, \\ \delta_\xi\psi_\alpha &= \sqrt{2}F\xi_\alpha - i\sqrt{2}\partial_\mu A\sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}, \\ \delta_\xi F &= i\sqrt{2}\partial_\mu\psi\sigma^\mu\bar{\xi}\ .\end{aligned} \tag{46}$$

The components A, ψ and F of a chiral multiplet transform into each other, so the multiplet is irreducible. Also, we see that $\theta\theta$-component of a chiral superfield transforms into itself plus a divergence term. $\theta\theta$ component of superfield is called F term.

An antichiral superfield $\Psi(x,\theta,\bar{\theta})$ is determined by the condition

$$D_\alpha\Psi(x,\theta,\bar{\theta}) = 0. \tag{47}$$

If $\Phi(x,\theta,\bar{\theta})$ is a chiral superfield, than $\Phi^\dagger(x,\theta,\bar{\theta})$ is antichiral. An antichiral superfield can be expressed in $y^\dagger = x^\mu + i(\theta\sigma^\mu\bar{\theta})$ coordinate as follows

$$\bar{\Phi}(x,\theta,\bar{\theta}) = A^*(y^\dagger) + \sqrt{2}\bar{\theta}\bar{\psi}(y^\dagger) + \bar{\theta}\bar{\theta}F^*(y^\dagger)\ . \tag{48}$$

Rewriting the antichiral superfield in terms of $x, \theta, \bar{\theta}$ coordinates we find

$$\bar{\Phi}(x,\theta,\bar{\theta}) = A^* + \sqrt{2}\bar{\theta}\bar{\psi} + \bar{\theta}\bar{\theta}F^* + i\partial_\mu A^*\theta\sigma^\mu\bar{\theta} - \frac{i}{\sqrt{2}}(\bar{\theta}\bar{\theta})(\theta\sigma^\mu\partial_\mu\bar{\psi}) - \frac{1}{4}\Box A^*(\theta\theta)(\bar{\theta}\bar{\theta})\ . \tag{49}$$

3.3 Products of Chiral and Antichiral SF

If Φ_i and Φ_j are two chiral superfields, then their product $\Phi_i\Phi_j$ is again a chiral superfield since $\bar{D}_{\dot\alpha}(\Phi_i\Phi_j) = (\bar{D}_{\dot\alpha}\Phi_i)\Phi_j + \Phi_i\bar{D}_{\dot\alpha}\Phi_j = 0$. In components this product is

$$\begin{aligned} \Phi_i\Phi_j &= \Big(A_i(y) + \sqrt{2}\theta\psi_i(y) + \theta\theta F_i(y)\Big)\Big(A_j(y) + \sqrt{2}\theta\psi_j(y) + \theta\theta F_j(y)\Big) \\ &= A_iA_j(y) + \sqrt{2}\theta(A_i\psi_j + A_j\psi_i)(y) + \theta\theta(F_iA_j + F_jA_i - \psi_i\psi_j)(y) \ , \end{aligned} \tag{50}$$

where we apply (60). So, from two chiral multiplets we get another chiral multiplet with components

$$(A_i,\psi_i,F_i)\times(A_j,\psi_j,F_j) = (A_iA_j, A_i\psi_j + A_j\psi_i, A_iF_j + F_iA_j - \psi_i\psi_j) \ . \tag{51}$$

And triple product of chiral multiplets is again a chiral multiplet:

$$\begin{aligned} &(A_i,\psi_i,F_i)\times(A_j,\psi_j,F_j)\times(A_k,\psi_k,F_k) = \\ &(A_iA_j, A_i\psi_j + A_j\psi_i, A_iF_j + F_iA_j - \psi_i\psi_j)\times(A_k,\psi_k,F_k) \\ &= (A_iA_jA_k, A_i\psi_jA_k + A_j\psi_iA_k + A_iA_j\psi_k, A_iA_jF_k - A_i\psi_j\psi_k + \text{ciclic } ijk) \ . \end{aligned} \tag{52}$$

The product of a chiral and an anti-chiral superfied is

$$\begin{aligned} \bar\Phi\Phi &= A^*A + \sqrt{2}A^*\theta\psi + \sqrt{2}A\bar\theta\bar\psi + \theta\theta A^*F + \bar\theta\bar\theta F^*A + \theta\sigma^\mu\bar\theta(-iA^*\partial_\mu A + iA\partial_\mu A^* + \psi\sigma_\mu\bar\psi) \\ &+ \frac{i}{\sqrt{2}}A^*(\theta\theta)(\partial_\mu\psi\sigma^\mu\bar\theta) + \sqrt{2}(\theta\theta)(\bar\theta\bar\psi)F - \frac{i}{\sqrt{2}}(\theta\theta)(\psi\sigma^\mu\bar\theta)\partial_\mu A^* \\ &+ \frac{i}{\sqrt{2}}(\bar\theta\bar\theta)(\theta\sigma^\mu\bar\psi)\partial_\mu A + \sqrt{2}(\bar\theta\bar\theta)(\theta\psi)F^* - \frac{i}{\sqrt{2}}(\bar\theta\bar\theta)(\theta\sigma^\mu\partial_\mu\bar\psi)A \\ &+ (\theta\theta)(\bar\theta\bar\theta)\left(-\frac{1}{4}A^*\Box A - \frac{1}{4}A\Box A^* - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar\psi + F^*F + \frac{1}{2}\partial_\mu A^*\partial^\mu A + \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar\psi\right) \ . \end{aligned} \tag{53}$$

The product $\bar\Phi\Phi$ is neither a chiral nor the anti-chiral field, since it do not satisfy any of constrains introduce above. This superfield is a real field. Its highest component $\theta\theta\bar\theta\bar\theta$ is known as $D-$ term and it transforms as space–time derivative under SUSY.

3.4 Wess–Zumino model

We want to construct a Lagrangian that is SUSY invariant and renormalizable for chiral and anti-chiral fields. First, we consider only one superfield Φ. The $\bar\Phi\Phi|_{\theta\theta\bar\theta\bar\theta}$ term is a kinetic term

$$\begin{aligned} \bar\Phi\Phi|_{\theta\theta\bar\theta\bar\theta} &= -\tfrac{1}{4}A^*\Box A - \tfrac{1}{4}A\Box A^* - \tfrac{i}{2}\partial_\mu\psi\sigma^\mu\bar\psi + F^*F + \tfrac{1}{2}\partial_\mu A^*\partial^\mu A + \tfrac{i}{2}\psi\sigma^\mu\partial_\mu\bar\psi \\ &= -A^*\Box A + i\bar\psi\bar\sigma^\mu\partial_\mu\psi + F^*F \ , \end{aligned} \tag{54}$$

where we discard surface terms.

The term $\Phi\Phi|_{\theta\theta\bar\theta\bar\theta}$ is SUSY invariant. However, it is itself a derivative term, so it does not contribute to the Lagrangian. Consider therefore the SUSY invariant term $\Phi\Phi|_{\theta\theta}$:

$$\Phi\Phi|_{\theta\theta} = 2AF - \psi\psi \ . \tag{55}$$

There is one more renormalizable term, namely $\Phi^3|_{\theta\theta}$:

$$\Phi^3|_{\theta\theta} = 3A^2F - 3A\psi\psi \ . \tag{56}$$

The higher order terms in chiral fields are not renormalizable. Thus, SUSY invariant Lagrangian is given by

$$\mathcal{L} = \bar{\Phi}\Phi|_{\theta\theta\bar{\theta}\bar{\theta}} + \Big(\frac{1}{2}m\Phi^2|_{\theta\theta} + \frac{1}{3}\lambda\Phi^3|_{\theta\theta} + \text{h.c.}\Big) \ , \tag{57}$$

where m and λ are coupling constants is SUSY invariant. In terms of component fields, the Lagrangian reads:

$$\begin{aligned}\mathcal{L} &= -A^*\Box A + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^*F \\ &+ \frac{m}{2}(2AF - \psi\psi) + \frac{m}{2}(2A^*F^* - \bar{\psi}\bar{\psi}) + \lambda(A^2F - A\psi\psi + A^{*2}F^* - A^*\bar{\psi}\bar{\psi}) \ .\end{aligned} \tag{58}$$

This is the Lagrangian that is SUSY invariant and renormalizable. It is called Wess-Zumino model. This model was constructed in 1974 by J. Wess and B. Zumino. It was the first four dimensional supersymmetric model. The fields F and F^* are auxiliary fields, since Lagrangian does not contain their derivatives. These fields can be eliminated from Lagrangian using the equation of motion

$$F = -mA^* - \lambda A^{*2} \ .$$

In this way we obtain so called 'on-shell' Lagrangian:

$$\mathcal{L} = \partial_\mu A^*\partial^\mu A + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi - |mA + \lambda A^2|^2 - \frac{1}{2}m\psi\psi - \frac{1}{2}m\bar{\psi}\bar{\psi} - \lambda A\psi\psi - \lambda A_i^*\bar{\psi}\bar{\psi} \ . \tag{59}$$

More general, with more than one chiral superfield, Lagrangian is

$$\mathcal{L} = \bar{\Phi}_i\Phi_i\Big|_{\theta\theta\bar{\theta}\bar{\theta}} + (W[\Phi_i]|_{\theta\theta} + \text{h.c.}) \ , \tag{60}$$

where the superpotential W is an analytic function of the chiral superfields:

$$W[\Phi_i] = \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}\lambda_{ijk}\Phi_i\Phi_j\Phi_k \ . \tag{61}$$

The quantities m_{ij} and λ_{ijk} are some totally symmetric constant coefficients. Lagrangian in terms of component fields reads

$$\mathcal{L} = \partial_\mu A_i^*\partial^\mu A_i + i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_i + F_i^*F_i + \Big(m_{ij}A_iF_j - \frac{1}{2}m_{ij}\psi_i\psi_j + \lambda_{ijk}A_iA_jF_k - \lambda_{ijk}A_i\psi_j\psi_k + c.c\Big) \ . \tag{62}$$

The previous Lagrangian gives the following equations of motion

$$\begin{aligned}&\Box A_i = m_{ij}F_j^* + 2\lambda_{ijk}A_j^*F_k^* - \lambda_{ijk}\bar{\psi}_j\bar{\psi}_k \\ &i\bar{\sigma}^\mu\partial_\mu\psi_i = m_{ij}\bar{\psi}_j + 2\lambda_{ijk}A_j\psi_k \\ &F_i = -m_{ij}A_j^* - \lambda_{ijk}A_j^*A_k^*.\end{aligned} \tag{63}$$

The auxiliary fields F_i can be integrated out using equations of motion (63). In that way we obtain

$$\begin{aligned}\mathcal{L} &= \partial_\mu A_i^* \partial^\mu A_i + i\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i - |m_{ij}A_j + \lambda_{ijk}A_jA_k|^2 \\ &- \frac{1}{2}m_{ij}\psi_i\psi_j - \frac{1}{2}m_{ij}\bar{\psi}_i\bar{\psi}_j - \lambda_{ijk}A_i\psi_j\psi_k - \lambda_{ijk}A_i^*\bar{\psi}_j\bar{\psi}_k \ . \end{aligned} \tag{64}$$

The last two terms are Yukawa interaction and we can define effective potential as:

$$V \equiv |m_{ij}A_j + \lambda_{ijk}A_jA_k|^2 = |F_i|^2 \ . \tag{65}$$

3.5 Vector superfield

An irreducble multiplet can be obtained imposing the reality condition

$$V^\dagger(x,\theta,\bar{\theta}) = V(x,\theta,\bar{\theta}). \tag{66}$$

This superfield is co–called the vector superfield. It has the following form

$$\begin{aligned} V(x,\theta,\bar{\theta}) &= C(x) + \sqrt{2}\theta\chi(x) + \sqrt{2}\bar{\theta}\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}M^*(x) + \theta\sigma^\mu\bar{\theta}v_\mu(x) \\ &+ (\theta\theta)\bar{\theta}\Big(\bar{\lambda}(x) - \frac{i}{\sqrt{2}}\bar{\sigma}^\mu\partial_\mu\chi(x)\Big) + (\bar{\theta}\bar{\theta})\theta\Big(\lambda(x) - \frac{i}{\sqrt{2}}\sigma^\mu\partial_\mu\bar{\chi}(x)\Big) \\ &+ \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\Big(D(x) - \frac{1}{2}\Box C(x)\Big) \ . \end{aligned} \tag{67}$$

The reality condition implies $C^* = C, v_\mu^* = v_\mu$ and $D^* = D$. A vector multiplet consists of 8 bosonic and 8 fermionic degrees of freedom.

The canonical dimension of a vector field is one, $[v_\mu] = 1$. Knowing $[\theta] = [\bar{\theta}] = -\frac{1}{2}$ implies

$$[V] = 0, \tag{68}$$

and subsequently

$$[C] = 0, \qquad [\chi] = [\bar{\chi}] = \frac{1}{2}, \qquad [M] = [M^*] = 1, \qquad [\lambda] = [\bar{\lambda}] = \frac{3}{2}, \qquad [D] = 2. \tag{69}$$

Let $i\Lambda$ be a chiral SF:

$$i\Lambda = f + \sqrt{2}\theta\varphi + \theta\theta F - i(\theta\sigma^\mu\bar{\theta})\partial_\mu f - \frac{i}{\sqrt{2}}\theta\theta(\bar{\theta}\bar{\sigma}^\mu\partial_\mu\varphi) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\Box f \ . \tag{70}$$

The following combination is a vector SF:

$$\begin{aligned} i\Lambda - i\Lambda^\dagger &= 2\mathrm{Re}f + \sqrt{2}\theta\varphi + \sqrt{2}\bar{\theta}\bar{\varphi} + \theta\theta F + \bar{\theta}\bar{\theta}F^* + 2(\theta\sigma^\mu\bar{\theta})\partial_\mu \mathrm{Im}f \\ &- \frac{i}{\sqrt{2}}(\theta\theta)(\bar{\theta}\bar{\sigma}^\mu\partial_\mu\varphi) - \frac{i}{\sqrt{2}}(\bar{\theta}\bar{\theta})(\theta\sigma^\mu\partial_\mu\bar{\varphi}) - \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\Box\mathrm{Re}f. \end{aligned} \tag{71}$$

We define a super-gauge transformation for the vector superfield in the following way:

$$V'(x,\theta,\bar{\theta}) = V(x,\theta,\bar{\theta}) + i(\Lambda(x,\theta,\bar{\theta}) - \Lambda^\dagger(x,\theta,\bar{\theta})), \tag{72}$$

where $\Lambda(x,\theta,\bar\theta)$ is a chiral superfield.

Under the super–gauge transformations the components of a vector superfield transform as

$$\begin{aligned}
C &\longrightarrow C + 2\mathrm{Re}f,\\
\chi &\longrightarrow \chi + \varphi,\\
M &\longrightarrow M + F,\\
v_\mu &\longrightarrow v_\mu + \partial_\mu(2\mathrm{Im}f),\\
\lambda &\longrightarrow \lambda,\\
D &\longrightarrow D.
\end{aligned} \tag{73}$$

The fields λ and D are gauge invariant, while v_μ transforms as the usual gauge field with $U(1)$ gauge parameter $2\mathrm{Im}f$. This means that the super-gauge symmetry is larger then usual gauge symmetry.

The super-gauge symmetry can be fixed by the following choice of parameters

$$-2\mathrm{Re}f = C, \qquad \varphi = -\chi, \qquad F = -M\ . \tag{74}$$

This gauge choice is known as Wess-Zumino gauge. In the WZ gauge a vector superfield is given by

$$V_{\mathrm{WZ}} = \theta\sigma^\mu\bar\theta v_\mu + (\theta\theta)(\bar\theta\bar\lambda) + (\bar\theta\bar\theta)(\theta\lambda) + \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)D\ . \tag{75}$$

Supersymmetry does not preserve WZ gauge, i.e. a supersymmetric transformation of a vector superfield in Wess-Zumino gauge gives a superfield which is not in this gauge. However, a suitable combination of supersymmetry transformation followed by a super–gauge transformation preserves the Wess-Zumino gauge. The fields in the vector multiplet are: a vector gauge field v_μ, a gaugino, λ_α and an auxiliary field, D.

We can perform further transformations that leave us within the WZ gauge with

$$i\Lambda = i\mathrm{Im}f + \theta\sigma^\mu\bar\theta\partial_\mu\mathrm{Im}f - \frac{i}{4}(\theta\theta)(\bar\theta\bar\theta)\Box\mathrm{Im}f. \tag{76}$$

This transformation will not change the conditions $C = \chi = M = 0$, and gives

$$v_\mu \longrightarrow v_\mu + \partial_\mu(2\mathrm{Im}f), \qquad \lambda \longrightarrow \lambda, \qquad D \longrightarrow D\ , \tag{77}$$

which is just the ordinary $U(1)$ gauge transformation.

It easy to prove that

$$V^2_{WZ}(x,\theta,\bar\theta) = \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)v_\mu(x)v^\mu(x) \tag{78}$$

$$V^n_{WZ}(x,\theta,\bar\theta) = 0,\ n \geq 3\ . \tag{79}$$

So, we find

$$e^{V_{\mathrm{WZ}}} = 1 + V_{\mathrm{WZ}} + \frac{1}{2}V^2_{\mathrm{WZ}} = 1 + \theta\sigma^\mu\bar\theta v_\mu + (\theta\theta)(\bar\theta\bar\lambda) + (\bar\theta\bar\theta)(\theta\lambda) + \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)\left(D + \frac{1}{2}v^\mu v_\mu\right)\ . \tag{80}$$

3.6 Lagrangian for Abelian gauge theory

In order to construct a kinetic term for a vector superfield, we have to find a supersymmetric analog of the field strength. Abelian field strength superfields are

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V,\ \bar{W}_{\dot{\alpha}} = -\frac{1}{4}D^2\bar{D}_{\dot{\alpha}}V \ . \tag{81}$$

One can check that both W_α and $\bar{W}_{\dot{\alpha}}$ are gauge invariant quantities:

$$\begin{aligned}
W_\alpha &= -\frac{1}{4}\bar{D}^2 D_\alpha V \longrightarrow -\frac{1}{4}\bar{D}^2 D_\alpha (V + i\Lambda - i\Lambda^\dagger) \\
&= W_\alpha - \frac{i}{4}\bar{D}^2 D_\alpha \Lambda \\
&= W_\alpha - \frac{i}{4}\bar{D}_{\dot{\beta}}\bar{D}^{\dot{\beta}} D_\alpha \Lambda \\
&= W_\alpha - \frac{i}{4}\bar{D}_{\dot{\beta}}\left(\{\bar{D}^{\dot{\beta}}, D_\alpha\} - D_\alpha \bar{D}^{\dot{\beta}}\right)\Lambda \\
&= W_\alpha + \frac{i}{4}\bar{D}^{\dot{\beta}}\{\bar{D}_{\dot{\beta}}, D_\alpha\}\Lambda \\
&= W_\alpha + \frac{i}{4}\bar{D}^{\dot{\beta}}\left(2i(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu \Lambda\right) \\
&= W_\alpha - \frac{1}{2}(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu \bar{D}^{\dot{\beta}}\Lambda \\
&= W_\alpha
\end{aligned} \tag{82}$$

and satisfy $\bar{D}_{\dot{\alpha}} W_\alpha = D_\alpha \bar{W}_{\dot{\alpha}} = 0$. W_α and $\bar{W}_{\dot{\alpha}}$ are a chiral and an antichiral spinorial superfields. It can be proven that

$$W_\alpha = \lambda_\alpha(y) + \theta_\alpha D(y) + i(\theta\theta)\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \bar{\lambda}^{\dot{\alpha}}(y) - (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) \ , \tag{83}$$

$$\bar{W}_{\dot{\alpha}} = \bar{\lambda}_{\dot{\alpha}}(y^\dagger) + \bar{\theta}_{\dot{\alpha}} D(y^\dagger) - \epsilon_{\dot{\alpha}\dot{\beta}}(\bar{\sigma}^{\mu\nu}\bar{\theta})^{\dot{\beta}} F_{\mu\nu}(y^\dagger) - i(\bar{\theta}\bar{\theta})(\partial_\mu \lambda(y^\dagger)\sigma^\mu)_{\dot{\alpha}} \ , \tag{84}$$

where $F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$. It can be shown that

$$W^\alpha W_\alpha\Big|_{\theta\theta} = i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - i\partial_\mu\bar{\lambda}\bar{\sigma}^\mu\lambda + D^2 - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + i\frac{1}{2}F^{\mu\nu}\widetilde{F}_{\mu\nu} \ , \tag{85}$$

where $\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is the dual tensor of $F_{\mu\nu}$. The last term in (85) is a total derivative. Lagrangian for the pure super Abelian gauge theory is given by

$$\mathcal{L} = \frac{1}{4}\left(W^\alpha W_\alpha\Big|_{\theta\theta} + \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\Big|_{\bar{\theta}\bar{\theta}}\right) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{2}D^2, \tag{86}$$

up to a total derivative term. The first term in (86) is Maxwell action.

Next, let us consider a coupling of the Abelian gauge superfield, V to chiral superfields, Φ_i Under the Abelian super-gauge transformation the chiral and antichiral superfields transform as

$$\Phi_i' = e^{-2igt_i\Lambda}\Phi_i, \tag{87}$$

$$\bar{\Phi}' = e^{2igt_i\Lambda^\dagger}\bar{\Phi}_i, \tag{88}$$

where t_i is (real) charge of superfield Φ_i. The term $\bar{\Phi}_i\Phi_i$ is not super–gauge invariant, but the term $\bar{\Phi}_i e^{2gt_iV}\Phi_i$ is:

$$\bar{\Phi}_i e^{2gt_iV}\Phi_i \to \bar{\Phi}_i e^{2igt_i\Lambda^\dagger(z)} e^{2gt_i(V+i\Lambda-i\Lambda^\dagger)} e^{-2igt_i\Lambda(z)}\Phi_i \tag{89}$$

$$= \bar{\Phi}_i e^{2gt_iV}\Phi_i \ . \tag{90}$$

In the WZ gauge only three terms in expression $\bar{\Phi}_i e^{2gt_iV}\Phi_i$ do not vanish:

$$\bar{\Phi}_i e^{2gt_iV}\Phi_i|_{WZ} = \bar{\Phi}_i\Phi_i + 2gt_i\bar{\Phi}_i V_{WZ}\Phi_i + 2g^2t_i^2\bar{\Phi}_i V_{WZ}^2\Phi_i \ . \tag{91}$$

It can be shown that the highest component of $\bar{\Phi}_i e^{2gt_iV}\Phi_i$ is given (up to total derivative terms) by

$$\begin{aligned}\bar{\Phi}_i e^{2gt_iV_{WZ}}\Phi_i|_{\theta\theta\bar{\theta}\bar{\theta}} &= F_i^*F_i + (\mathcal{D}_{\mu i}A_i)(\mathcal{D}_i^{\mu\dagger}A_i^*) + i\psi_i\sigma^\mu\mathcal{D}_{\mu i}^\dagger\bar{\psi}_i \\ &+ gt_iD|A_i|^2 - g\sqrt{2}[(\bar{\psi}_i\bar{\lambda})A_i + (\psi_i\lambda)A_i^*] \ ,\end{aligned} \tag{92}$$

where we introduce covariant derivatives as

$$\mathcal{D}_{\mu i}A_i = (\partial_\mu + igt_iv_\mu)A_i \tag{93}$$

$$\mathcal{D}_{\mu i}\psi_i = (\partial_\mu + igt_iv_\mu)\psi_i \ . \tag{94}$$

The action is

$$S = \int d^4x\left(\bar{\Phi}_i e^{2gt_iV}\Phi_i|_{\theta\theta\bar{\theta}\bar{\theta}} + \frac{1}{4}W^\alpha W_\alpha|_{\theta\theta} + \frac{1}{4}\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}|_{\bar{\theta}\bar{\theta}} + W[\Phi_i]|_{\theta\theta} + \bar{W}[\bar{\Phi}_i]|_{\bar{\theta}\bar{\theta}}\right) \ . \tag{95}$$

The superpotential $W(\Phi)$ has to respect super-gauge symmetry. Now, we can write down the full Lagrangian in components forms

$$\begin{aligned}\mathcal{L} \ &= F_i^*F_i + (\mathcal{D}_i^\mu A_i)^*(\mathcal{D}_{\mu i}A_i) + i\psi_i\sigma^\mu\mathcal{D}_{\mu i}^\dagger\bar{\psi}_i + gt_iDA_i^*A_i - \sqrt{2}gt_i[(\bar{\psi}_i\bar{\lambda})A_i + (\psi_i\lambda)A_i^*] \\ &\quad + \tfrac{1}{2}D^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda\sigma^\mu\partial_\mu\bar{\lambda} \\ &\quad + F_i\tfrac{\partial W}{\partial A_i} - \tfrac{1}{2}\tfrac{\partial^2 W}{\partial A_i\partial A_j}\psi_i\psi_j + F_i^*\tfrac{\partial\bar{W}}{\partial A_i^*} - \tfrac{1}{2}\tfrac{\partial^2\bar{W}}{\partial A_i^*\partial A_j^*}\bar{\psi}_i\bar{\psi}_j \ .\end{aligned} \tag{96}$$

Here, we can use the equations of motion to integrate out auxiliary fields:

$$\begin{aligned}F_i &= -\frac{\partial\bar{W}}{\partial A_i^*} \\ F_i^* &= -\frac{\partial W}{\partial A_i} \\ D &= -gt_iA_i^*A_i \ .\end{aligned} \tag{97}$$

Substituting these results in (96) we obtain the on-shell Lagrangian:

$$\begin{aligned}\mathcal{L} \ &= -\left|\tfrac{\partial W}{\partial\varphi_i}\right|^2 + (\mathcal{D}_i^\mu A_i)^\dagger(\mathcal{D}_{\mu i}A_i) + i\psi_i\sigma^\mu\mathcal{D}_{\mu i}^\dagger\bar{\psi}_i - \tfrac{1}{2}(gt_iA_i^*A_i)^2 - \sqrt{2}gt_i[(\bar{\psi}_i\bar{\lambda})A_i + (\psi_i\lambda)A_i^*] \\ &\quad - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \tfrac{1}{2}\tfrac{\partial^2 W}{\partial A_i\partial A_j}\psi_i\psi_j - \tfrac{1}{2}\tfrac{\partial^2\bar{W}}{\partial A_i^*\partial A_j^*}\bar{\psi}_i\bar{\psi}_j \ .\end{aligned} \tag{98}$$

In this theory A is a scalar, λ is a gaugino and ψ is a chiral fermion.

3.7 Super Quantum Electrodynamics

Super QED is a supersymmetric extension of the standard QED. Let $\Phi_+ = (A_+, \psi_+, F_+)$ and $\Phi_- = (A_-, \psi_-, \Phi_-)$ be two chiral superfields with charges q and $-q$. Under $U(1)$ super–gauge transformations they transform as follows

$$\Phi_+ \to \Phi'_+ = e^{-2iq\Lambda(z)}\Phi_+ \ , \tag{99}$$

$$\Phi_- \to \Phi'_- = e^{2iq\Lambda(z)}\Phi_- \ . \tag{100}$$

The action is given by

$$\begin{aligned} S &= \int d^8z\Big(\bar\Phi_+ e^{2qV}\Phi_+ + \bar\Phi_- e^{-2qV}\Phi_- + \delta^{(2)}(\bar\theta)(\frac{1}{4}W^\alpha W_\alpha + m\Phi_+\Phi_-) \\ &+ \delta^{(2)}(\theta)\left(\frac{1}{4}\bar W_{\dot\alpha}\bar W_{\dot\alpha} + m\bar\Phi_+\bar\Phi_-\right)\Big) \ . \end{aligned} \tag{101}$$

Note that the mass terms m$\Phi_\pm\Phi_\pm$ are absent since they are not the super gauge invariant. The Lagrangian in component form is given by

$$\begin{aligned} \mathcal{L} = &|F_+|^2 + |\mathcal{D}_\mu A_+|^2 + i\psi_+\sigma^\mu \mathcal{D}^\dagger_\mu \bar\psi_+ - \sqrt{2}q[(\psi_+\lambda)A^*_+ + \text{h.c.}] + qA^*_+A_+D \\ &+ |F_-|^2 + |\mathcal{D}_\mu A_-|^2 + i\psi_-\sigma^\mu \mathcal{D}_\mu \bar\psi_- + \sqrt{2}q[(\psi_-\lambda)A^*_- + \text{h.c.}] - qA^*_-A_-D \\ &+ \frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda\sigma^\mu\partial_\mu\bar\lambda + m(A_+F_- + A_-F_+ - \psi_+\psi_-) + m(A^*_+F^*_- + A^*_-F^*_+ - \bar\psi_+\bar\psi_-) \ . \end{aligned} \tag{102}$$

Substituting the equations of motion

$$F_\pm = -mA^*_\mp \tag{103}$$

$$D = -qA^*_+A_+ + qA^*_-A_- \tag{104}$$

into Lagrangian in (102) we obtain the on-shell Lagrangian:

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda\sigma^\mu\partial_\mu\bar\lambda + i\psi_+\sigma^\mu\mathcal{D}^\dagger_\mu\bar\psi_+ + i\psi_-\sigma^\mu\mathcal{D}_\mu\bar\psi_- + |\mathcal{D}_\mu A_+|^2 + |\mathcal{D}_\mu A_-|^2 \\ &- \frac{q}{2}(A^*_+A_+A^*_-A_-) - m^2(A^*_+A_+\varphi^*_-A_-) - m(\psi_+\psi_- + \bar\psi_+\bar\psi_-) \\ &- \sqrt{2}q[(\psi_+\lambda)A^*_+ + (\bar\psi_+\bar\lambda)A_+ - (\psi_-\lambda)A^*_- - (\bar\psi_-\bar\lambda)A_-] \ , \end{aligned} \tag{105}$$

where

$$\mathcal{D} = \partial_\mu + iqA_\mu \ . \tag{106}$$

We introduce Dirac spinor for matter fields:

$$\Psi = \begin{pmatrix} (\psi_+)_\alpha \\ (\bar\psi_-)^{\dot\alpha} \end{pmatrix} \tag{107}$$

and the Majorana spinor for gaugino field

$$\lambda_{\rm M} = \left(\begin{array}{c} \lambda_\alpha \\ \bar\lambda^{\dot\alpha} \end{array} \right) \ . \tag{108}$$

In terms of Dirac and Majorana spinors Lagrangian (105) reads

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\lambda_{\rm M}\gamma^\mu\partial_\mu\bar\lambda_{\rm M} + i\bar\Psi\gamma^\mu\mathcal{D}_\mu\Psi - m\bar\Psi\Psi \\ &+ |\mathcal{D}_\mu A_+|^2 + |\mathcal{D}_\mu A_-|^2 - m^2(|A_+|^2 + |A_-|^2) - \frac{q}{2}(|A_+|^2 + |A_-|^2)^2 \\ &- \sqrt{2}q(A_+^*\bar\lambda_{\rm M}\Psi_L - A_-\bar\lambda_{\rm M}\Psi_R + c.c) \ . \end{aligned} \tag{109}$$

3.8 Non-Abelian Super–Yang-Mills theories

In the previous section we discussed the supersymmetric $U(1)$ gauge theories. Now, we generalize the Abelian symmetry to the non–Abelian one. Let us consider a simple connected group G. The hermitian generators of G are T^a and they satisfy

$$[T^a, T^b] = if^{abc}T^c \ , \tag{110}$$

where f^{abc} are structure constants. Chiral superfields transform under rigid transformations as follows

$$\Phi_I = \exp(-ig\Lambda^a T^a)_{IJ}\Phi_J \ ,$$

where Λ^a are constant real parameters and g is a coupling constant. The kinetic term $\bar\Phi_I\Phi_I$ is invariant under the rigid transformations.

To localize this symmetry in the superspace, the parameters Λ^a have to become chiral functions. The superfields, Φ_I and $\bar\Phi_I$ transform under gauge transformation in the following way

$$\Phi'_I = \exp(-ig\Lambda)_{IJ}\Phi_J, \qquad \bar\Phi'_I = \bar\Phi_J \exp(-ig\Lambda^\dagger)_{JI}, \tag{111}$$

where $\Lambda = 2g\Lambda^a(x,\theta,\bar\theta)T^a, \Lambda^\dagger = 2g\Lambda^{a*}(x,\theta,\bar\theta)T^a$ are the chiral and antichiral superfields. The term $\bar\Phi_I\Phi_I$ is not gauge invariant. To recover invariance we introduce a vector (or gauge or Yang-Mills) superfield $V = 2gV^aT^a$. It is a Hermitian matrix transforming under the adjoint representation of G. In order for the term $\bar\Phi_I(e^V)_{IJ}\Phi_J$ to be invariant under super-gauge transformation, the super–gauge transformation of the vector superfield has to be given by

$$e^{+V'} = e^{-i\Lambda^\dagger}e^{+V}e^{+i\Lambda} \ . \tag{112}$$

Writing $V' = V + \delta V$, where δV is an infinitesimal change of vector superfield in the first order in Λ, we obtain

$$e^{V+\delta V} - e^V = \delta V + \frac{1}{2}(V\delta V + \delta V V) + \frac{1}{6}(\delta V V^2 + V\delta V V + V^2\delta V) + \dots \ . \tag{113}$$

On the other hand, from (112) it follows

$$e^{V+\delta V} - e^V = e^V i\Lambda - i\Lambda^\dagger e^V = i\Lambda - i\Lambda^\dagger + Vi\Lambda - i\Lambda^\dagger V + \frac{i}{2}V^2\Lambda - \frac{i}{2}\Lambda^\dagger V^2 + \dots \ . \tag{114}$$

To solve these equations we assume that δV is a expansion in powers of V

$$\delta V = \delta V^{(0)} + \delta V^{(1)} + \delta V^{(2)} + \dots \ . \tag{115}$$

The term $\delta V^{(n)}$ is $n-$th order in power of the vector superfield.

Comparing (113) and (114) we find

$$\begin{aligned}
\delta V^{(0)} &= i(\Lambda - \Lambda^\dagger), \\
\delta V^{(1)} &= \frac{i}{2}[V, \Lambda + \Lambda^\dagger], \\
\delta V^{(2)} &= \frac{i}{12}[V, [V, \Lambda - \Lambda^\dagger]] \ .
\end{aligned} \tag{116}$$

Collecting these terms together we obtain

$$\delta V = i(\Lambda - \Lambda^\dagger) + \frac{i}{2}[V, \Lambda + \Lambda^\dagger] + \frac{i}{12}[V, [V, \Lambda - \Lambda^\dagger]] + \dots \ , \tag{117}$$

or in component notation

$$\delta V^a = i(\Lambda - \Lambda^*)^a - g f^{abc} V^b (\Lambda + \Lambda^*)^c - \frac{i}{3} g^2 f^{dbc} f^{aed} V^e V^b (\Lambda - \Lambda^*)^c + O(V^3) \ . \tag{118}$$

It can be shown that we can impose the Wess–Zumino gauge

$$V_{\rm WZ} = (\theta\sigma^\mu\bar\theta) v_\mu + (\theta\theta)(\theta\lambda) + (\bar\theta\bar\theta)(\theta\lambda) + \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta) D \ . \tag{119}$$

We use the notation $v_\mu = 2g v^a_\mu T^a, \ D = 2g D^a T^a \dots$ where v^a_μ are the gauge fields. The Wess-Zumino gauge does not fix super–gauge symmetry completely. The super–gauge transformations determined by

$$i\Lambda = i{\rm Im} f + (\theta\sigma^\mu\bar\theta)\partial_\mu {\rm Im} f - \frac{i}{4}(\theta\theta)(\bar\theta\bar\theta){\rm Im} f \tag{120}$$

preserve WZ gauge. This residual symmetry corresponds to usual gauge symmetry. The vector superfield transformation is given by:

$$\delta V_{\rm WZ} = i(\Lambda - \Lambda^\dagger) + \frac{i}{2}[V_{\rm WZ}, \Lambda + \Lambda^\dagger] \ . \tag{121}$$

To find the Lagrangian for super Yang–Mills fields we introduce two new superfields

$$W_\alpha = -\frac{1}{4}\bar D^2 \left(e^{-V} D_\alpha e^{+V}\right), \qquad \bar W_{\dot\alpha} = +\frac{1}{4} D^2 \left(e^{+V} \bar D_{\dot\alpha} e^{-V}\right) \ . \tag{122}$$

It is clear that W_α and $\bar W_{\dot\alpha}$ are chiral and antichiral superfields, respectively. The superfield strength W_α can be expanded in V:

$$W_\alpha = -\frac{1}{4}\bar D^2 D_\alpha V - \frac{1}{8}\bar D^2 [D_\alpha V, V] - \frac{1}{24}\bar D^2 [[D_\alpha V, V], V] + \dots \ . \tag{123}$$

In the Abelian case this expression reduces to $W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V$, as we expected. It can be shown that W_α transforms covariantly under the super–gauge transformations:

$$W_\alpha \longrightarrow e^{-i\Lambda} W_\alpha e^{+i\Lambda} \ . \tag{124}$$

In the Wess-Zumino gauge we have

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V_{\mathrm{WZ}} - \frac{1}{8}\bar{D}^2 [D_\alpha V_{\mathrm{WZ}}, V_{\mathrm{WZ}}] \ . \tag{125}$$

Finally, we arrive at $W_\alpha = 2gW^a_\alpha T^a$, where

$$W^a_\alpha = \lambda^a_\alpha(y) + \theta_\alpha D^a(y) + i\theta\theta(\sigma^\mu \mathcal{D}_\mu \bar{\lambda}^a(y))_\alpha - (\sigma^{\mu\nu}\theta)_\alpha F^a_{\mu\nu}(y) \ . \tag{126}$$

The term $\mathrm{Tr}(W_\alpha W^\alpha)$ is super–gauge and Lorentz invariant. Its $\theta\theta$ component is invariant under supersymmetry. Therefore, the Lagrangian for the super-non-Abelian gauge field is

$$\mathcal{L} = \frac{1}{16g^2k}\mathrm{Tr}\left(W^\alpha W_\alpha\Big|_{\theta\theta} + \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\Big|_{\bar{\theta}\bar{\theta}} \right) \Big) \tag{127}$$

$$= -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + i\lambda^a\sigma^\mu(\mathcal{D}_\mu\bar{\lambda})^a + \frac{1}{2}D^a D^a, \tag{128}$$

where

$$\begin{aligned}(\mathcal{D}_\mu\bar{\lambda})^a &= \partial_\mu\bar{\lambda}^a - gf^{abc}v^b_\mu\bar{\lambda}^c \ , \\ F^a_{\mu\nu} &= \partial_\mu v^a_\nu - \partial_\nu v^a_\mu - gf^{abc}v^b_m v^c_\nu \ ,\end{aligned}$$

and the constant k is determined by $\mathrm{Tr}(T^aT^b) = \frac{1}{2}k\delta^{ab}$.

Lagrangian (128) describes super-Yang-Mills theory without matter fields. The Lagrangian for matter sector is given by

$$\mathcal{L}_{\mathrm{matter}} = \bar{\Phi}e^V\Phi\Big|_{(\theta\theta)(\bar{\theta}\bar{\theta})} + W(\Phi)\Big|_{\theta\theta} + \bar{W}(\bar{\Phi})\Big|_{\bar{\theta}\bar{\theta}} \ , \tag{129}$$

where the superpotential $W(\Phi)$ is a gauge invariant quantity i.e. it belongs to a singlet representation of G.

In the Wess-Zumino gauge the kinetic term for matter fields, $\bar{\Phi}e^V\Phi = \bar{\Phi}_I(e^V)_{IJ}\Phi_J$ is given by

$$\bar{\Phi}e^V\Phi = \bar{\Phi}\Phi + \bar{\Phi}V\Phi + \frac{1}{2}\bar{\Phi}V^2\Phi. \tag{130}$$

Its D-component is

$$\bar{\Phi}e^V\Phi\Big|_{(\theta\theta)(\bar{\theta}\bar{\theta})} = F^*F + (\mathcal{D}^\mu A)^\dagger(\mathcal{D}_\mu A) - i\bar{\psi}\bar{\sigma}^\mu\mathcal{D}_\mu\psi + gA^*DA - \sqrt{2}gA^*(\psi\lambda) - \sqrt{2}g(\bar{\psi}\bar{\lambda})A, \tag{131}$$

where:

$$\mathcal{D}_\mu A = (\partial_\mu + igv^a_\mu T^a)A, \qquad \mathcal{D}_\mu\psi = (\partial_\mu + igv^a_\mu T^a)\psi. \tag{132}$$

Let us stress that the fields A, ψ and F transform under fundamental, while D, λ and v_μ under adjoint representation of G. Adding all terms together we obtain the off-shell Lagrangian:

$$\begin{aligned}\mathcal{L} = \quad & -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - i\bar{\lambda}^a\bar{\sigma}^\mu\mathcal{D}_\mu\lambda^a + (\mathcal{D}_\mu A)^\dagger(\mathcal{D}^\mu A) - i\bar{\psi}\bar{\sigma}^\mu\mathcal{D}_\mu\psi \\ & +gA^\dagger D^a T^a A - \sqrt{2}g\psi_I\lambda^a T^a_{IJ}A^*_J - \sqrt{2}g\bar{\lambda}^a\bar{\psi}_I T^a_{IJ}A_J \\ & + \left(F_I\tfrac{\partial W}{\partial A_I} - \tfrac{1}{2}\tfrac{\partial^2 W}{\partial A_I \partial A_J}\psi_I\psi_J + h.c.\right) \\ & +\tfrac{1}{2}D^a D^a + F^*_I F_I \ . \end{aligned} \tag{133}$$

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