Лоренцова група

Дефинишући матрицу $X = \vec{\sigma}\vec{x}$, где су σ_i Паулијеве матрице, а $\vec{x}=(x,y,z)$ дискутовати везу између група $SU(2)$ и $SO(3)$.

$SO(3)$ ротације у 3 димензије — чувају дужину вектора

$R \in SO(3) \qquad \vec{x}' = R\vec{x}$

$$|\vec{x}'|^2 = \vec{x}'^T \cdot \vec{x}' = (R\vec{x})^T R \cdot \vec{x} = \vec{x}^T \underbrace{R^T R}_{1} \vec{x} = \vec{x}^T \vec{x} = |\vec{x}|^2 \qquad R^T R = 1$$

$\det R = \pm 1$ $O(3)$ ако се укључе само матрице са јединичном детерминантом $SO(3)$ група. 3×3 матрице $\det R = 1$, $R^T R = 1$.

вектор $\vec{x}$ преводимо у хермитски оператор $X = \vec{\sigma}\vec{x} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$

$\det X = -|\vec{x}|^2$

На језику хермитских оператора ротације су трансф. које чувају детерминанту.

$\underline{X' = U X U^+}$ $\qquad \det X' = \det U \cdot \det U^+ \det X$

$U \in SU(2)$: $UU^+ = 1$; $\det U = 1$ $\qquad$ 2×2 комплексне матрице

Произвољна $SU(2)$ трансф. има облик: $U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, $|a|^2+|b|^2 = 1$, $a, b \in \mathbb{C}$

Простор параметара $SU(2)$ групе $\boxed{|a|^2+|b|^2 = a_1^2+a_2^2+b_1^2+b_2^2 = 1}$ сфера у 4 дим.

• Како се са хермитске матрице X прелази на вектор $\vec{x}$?

$X = \vec{\sigma}\vec{x} = \sigma_i x_i \quad ; \quad \mathrm{tr}\, \sigma_i \sigma_j = 2\delta_{ij}$

$$\mathrm{tr}(\sigma_j X) = \mathrm{tr}(\sigma_j \sigma_i) x_i = 2\delta_{ij} x_i = 2 x_j \Rightarrow \boxed{x_i = \frac{1}{2}\mathrm{tr}(\sigma_i X)}$$

• За дато U наћи R

$$x_i' = R_{ij} x_j = \frac{1}{2}\mathrm{tr}(\sigma_i X') = \frac{1}{2}\mathrm{tr}(\sigma_i U X U^+) = \frac{1}{2}\mathrm{tr}(\sigma_i U \sigma_j x_j U^+) = \frac{1}{2}\mathrm{tr}(\sigma_i U \sigma_j U^+) x_j$$

$$\Rightarrow \boxed{R_{ij} = \frac{1}{2}\mathrm{tr}(\sigma_i U \sigma_j U^+)}$$

И U и $-U$ дају исто R_{ij} — $SU(2)$ је универзално наткривајућа од $SO(3)$.

пример: $U = \cos\frac{\varphi}{2} \hat{I} + i(\vec{\sigma}\vec{n})\sin\frac{\varphi}{2}$ ово је $SU(2)$ матрица која представља ротацију.

$$U^+ = \cos\frac{\varphi}{2} I - i(\vec{\sigma}\vec{n})\sin\frac{\varphi}{2} \qquad UU^+ = \cos^2\frac{\varphi}{2} I \cancel{- i\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\vec{\sigma}\vec{n}} \cancel{+ i\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\vec{\sigma}\vec{n}} + (\vec{\sigma}\vec{n})^2 \sin^2\frac{\varphi}{2}$$

$$UU^+ = \cos^2\frac{\varphi}{2} I + \sin^2\frac{\varphi}{2}\, \sigma_i n_i \sigma_j n_j = \cos^2\frac{\varphi}{2} I + \sin^2\frac{\varphi}{2}(\delta_{ij} + \underbrace{i\varepsilon_{ijk}\sigma_k}_{0}) n_i n_j = I$$

ротација око x осе $\quad U = \cos\frac{\varphi}{2} I + i\sin\frac{\varphi}{2}\sigma^1 = \begin{pmatrix} \cos\frac{\varphi}{2} & i\sin\frac{\varphi}{2} \\ i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}$

$$U = \begin{pmatrix} \cos\frac{\varphi}{2} & i\sin\frac{\varphi}{2} \\ i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}$$

$$X' = U X U^{+}$$

$$X' = \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix} = \begin{pmatrix} \cos\frac{\varphi}{2} & i\sin\frac{\varphi}{2} \\ i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix} \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \begin{pmatrix} \cos\frac{\varphi}{2} & -i\sin\frac{\varphi}{2} \\ -i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\varphi}{2} & i\sin\frac{\varphi}{2} \\ i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix} \begin{pmatrix} z\cos\frac{\varphi}{2} - i(x-iy)\sin\frac{\varphi}{2} & -iz\sin\frac{\varphi}{2} + (x-iy)\cos\frac{\varphi}{2} \\ (x+iy)\cos\frac{\varphi}{2} + iz\sin\frac{\varphi}{2} & -i(x+iy)\sin\frac{\varphi}{2} - z\cos\frac{\varphi}{2} \end{pmatrix}$$

$$= \begin{pmatrix} z\cos^2\frac{\varphi}{2} - i(\cancel{x}-iy)\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} + i(\cancel{x}+iy)\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} - z\sin^2\frac{\varphi}{2} & -iz\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} + (\cancel{x}-iy)\cos^2\frac{\varphi}{2} + (x+iy)\sin^2\frac{\varphi}{2} - iz\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} \\ iz\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} + (x+iy)\sin^2\frac{\varphi}{2} + (x+iy)\cos^2\frac{\varphi}{2} + iz\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} & z\sin^2\frac{\varphi}{2} + i(\cancel{x}+iy)\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} - i(\cancel{x}+iy)\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} - z\cos^2\frac{\varphi}{2} \end{pmatrix}$$

$$= \begin{pmatrix} z\cos\varphi - y\sin\varphi & -iz\sin\varphi + x - iy\cos\varphi \\ iz\sin\varphi + x + iy\cos\varphi & -z\cos\varphi + y\sin\varphi \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Ово је ротација око x-осе у пасивној репрезентацији.

1. а) Доказати да је $O(3)$ неповезана група.
б) Показати да је $SO(3)$ инваријантна подгрупа групе $O(3)$.

а) $M \in O(3) \Leftrightarrow MM^T = M^TM = I$, M - реална 3×3 матрица, производ је множење матрица,

$\det(MM^T) = (\det M)^2 = \det I = 1 \Rightarrow \det M = \pm1 \rightarrow$ постоје две класе повезаности: матрице са детерминантом $+1$ и -1, не могу се превести непрекидном трансформацијом једне у друге.

б) $SO(3)$ $MM^T = M^TM = I$ и $\det M = 1$, $SO(3) \subset O(3)$

затвореност: $M_1, M_2 \in SO(3)$ $(M_1M_2)^TM_1M_2 = M_2^TM_1^TM_1M_2 = I$ и $\det(M_1M_2) = \det M_1 \det M_2 = 1$

асоцијативност иначе важи за множење матрица.

неутрал $I \in SO(3)$ $\det I = 1$ и $I^TI = I$

инверз $\det M = 1$ ако $M \in SO(3)$ па је M инвертибилна матрица

инваријантност: $\forall S \in O(3)$ треба да важи $S^{-1}MS \in SO(3)$ ако $M \in SO(3)$

$(S^{-1}MS)^TS^{-1}MS = S^TM^T\underbrace{SS^T}_{1}MS = I$; $\det(S^{-1}M\cdot S) = \det S^{-1}\cdot\det M\cdot\det S = \underbrace{(\det S)^2}_{=1}\det M = \det M$

$S^{-1} = S^T$

Показати да је L_+ инваријантна ...

Лоренцова група: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$; $x'^2 = x^2$; $x^2 = g_{\mu\nu} x^\mu x^\nu$

$$x'^2 = g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma x^\rho x^\sigma = g_{\rho\sigma} x^\rho x^\sigma = x^2$$

$$\Rightarrow g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma} \quad \text{или} \quad (\Lambda^T)_\rho{}^\mu g_{\mu\nu}\Lambda^\nu{}_\sigma = g_{\rho\sigma} \Rightarrow \boxed{\Lambda^T g \Lambda = g}$$

$$\Lambda \in O(1,3,\mathbb{R}) \Leftrightarrow \Lambda^T g \Lambda = g$$

$$\det \Lambda^T g \Lambda = (\det\Lambda)^2 \cdot \det g = \det g \Rightarrow \det\Lambda = \pm 1$$

$$g_{00} = g_{\mu\nu}\Lambda^\mu{}_0 \Lambda^\nu{}_0 = g_{00}(\Lambda^0{}_0)^2 + g_{ii}(\Lambda^i{}_0)^2 = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3}(\Lambda^i{}_0)^2 = 1$$

$$(\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3}(\Lambda^i{}_0)^2 \geq 1 \Rightarrow \Lambda^0{}_0 \geq 1 \quad \text{или} \quad \Lambda^0{}_0 \leq -1$$

Постоје 4 класе повезаности у групи $O(1,3,\mathbb{R})$

косет	$\det\Lambda$	$\Lambda^0{}_0$	диск. трансф.	
$L_+^\uparrow$	1	≥ 1	I	
$L_-^\uparrow$	-1	≥ 1	$I_S = g = \mathrm{diag}(1,-1,-1,-1)$	← просторна инверзија
$L_+^\downarrow$	1	≤ -1	$I_T = -g = \mathrm{diag}(-1,1,1,1)$	← временска инверзија
$L_-^\downarrow$	-1	≤ -1	$I_{ST} = -I$	

$L_+^\uparrow$ је група: асоц. важи, $I \in L_+^\uparrow$, инверзи постоје јер су инверт. матрице

$\det\Lambda = 1 \Rightarrow \det\Lambda^{-1} = 1$ $\quad \Lambda^0{}_0 \geq 1 \Rightarrow (\Lambda^{-1})^0{}_0 \geq 1 \Rightarrow \Lambda^{-1} \in L_+^\uparrow$ (Ни код ротација ни код бустова се не мења $\Lambda^0{}_0$ при инвертовању)

затвореност: $\Lambda_1, \Lambda_2 \in L_+^\uparrow$ $\quad \det(\Lambda_1\Lambda_2) = \det\Lambda_1 \cdot \det\Lambda_2 = 1$

$$(\Lambda_1\Lambda_2)^0{}_0 = \underbrace{\Lambda_1{}^0{}_0 \Lambda_2{}^0{}_0}_{\geq 1} + \Lambda_1{}^0{}_i \Lambda_2{}^i{}_0$$

За две ротације $\Lambda_1{}^0{}_i = \Lambda_2{}^i{}_0 = 0$ па је $(\Lambda_1\Lambda_2)^0{}_0 = \Lambda_1{}^0{}_0 \Lambda_2{}^0{}_0 = 1$

за буст и ротацију $(\Lambda_1\Lambda_2)^0{}_0 = \Lambda_1{}^0{}_0 \Lambda_2{}^0{}_0 \geq 1$

за два буста дуж истог правца нпр. x осе $\Lambda_1 = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & & \\ -\beta_1\gamma_1 & \gamma_1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $\Lambda_2 = \begin{pmatrix} \gamma_2 & -\beta_2\gamma_2 & & \\ -\beta_2\gamma_2 & \gamma_2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$$(\Lambda_1\Lambda_2)^0{}_0 = \gamma_1\gamma_2 + \beta_1\beta_2\gamma_1\gamma_2 = \gamma_1\gamma_2(1+\beta_1\beta_2) = \frac{1}{\sqrt{(1-\beta_1^2)(1-\beta_2^2)}}(1+\beta_1\beta_2) \overset{?}{\geq} 1$$

$$\Leftrightarrow 1+\beta_1\beta_2 \geq \sqrt{1-\beta_1^2-\beta_2^2+\beta_1^2\beta_2^2} \;\Big|^2 \Leftrightarrow \cancel{1}+\cancel{\beta_1^2\beta_2^2}+2\beta_1\beta_2 \geq \cancel{1}-\beta_1^2-\beta_2^2+\cancel{\beta_1^2\beta_2^2} \Leftrightarrow \beta_1^2+\beta_2^2+2\beta_1\beta_2 \geq 0 \Leftrightarrow T$$

За два буста дуж различитих оса $(\Lambda_1\Lambda_2)^0{}_0 = \gamma_1\gamma_2 \geq 1$

Како се произвољно $\Lambda \in L_+^\uparrow$ може изразити као производ ротације и буста доказана је затвореност.

За доказ инваријантности $L_+^\uparrow$ довољно је показати да се конјугацијом са I_S и I_T добија елемент из $L_+^\uparrow$ јер:

Ако за $A, B \in O(1,3,\mathbb{R})$ важи $A\Lambda A^{-1} \in L_+^\uparrow$ и $B\Lambda B^{-1} \in L_+^\uparrow$ за свако $\Lambda \in L_+^\uparrow$ тада:

$(AB)\Lambda(AB)^{-1} = A(B\Lambda B^{-1})A^{-1} \in L_+^\uparrow$ за свако $\Lambda \in L_+^\uparrow$.

конјугација са $I_{P/T}$ $\det\left(I_{P/T}\Lambda I_{P/T}^{-1}\right) = \left(\det I_{P/T}\right)^2 \cdot \det\Lambda = \det\Lambda = 1$

$$\left(I_{P/T}\Lambda I_{P/T}^{-1}\right)^0{}_0 = I_{P/T}{}^0{}_0\, \Lambda^0{}_0 \left(I_{P/T}^{-1}\right)^0{}_0 = \Lambda^0{}_0\left(\left(I_{P/T}\right)^0{}_0\right)^2 = \Lambda^0{}_0 \geq 1$$

4. Одредити матричне елементе генератора Лоренцове групе у дефиниционој репрезентацији.

$U(\omega) = e^{-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}}$ произвољна репрезентација.

Дефинициона репрезентација: $(U(\omega))^\mu{}_\nu \cdot x^\nu = x'^\mu = \underbrace{\Lambda^\mu{}_\nu}_{\text{инфинитезимална Л.т.}} x^\nu = (\delta^\mu{}_\nu + \omega^\mu{}_\nu)x^\nu$

Ротације:

∘ око x осе за угао θ_1; $U(\theta_1) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_1 & \sin\theta_1 \\ & & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \approx I_4 + \begin{bmatrix} & & & \\ & & & \\ & & & \theta_1 \\ & & -\theta_1 & \end{bmatrix} \Rightarrow \omega^2{}_3 = -\omega^3{}_2 = \theta_1$

$\omega^{23} = -\theta_1$

$U(\theta_1) = e^{-\frac{i}{2}(\omega^{23}M_{23} + \omega^{32}M_{32})} = e^{-i\omega^{23}M_{23}} = e^{i\theta_1 M_{23}} \Rightarrow M_{23} = -i\left.\frac{dU(\theta_1)}{d\theta_1}\right|_{\theta_1=0}$

$M_{23} = \begin{bmatrix} & & & \\ & & & \\ & & & -i \\ & & i & \end{bmatrix}$

∘ око y осе за угао θ_2; $U(\theta_2) = \begin{bmatrix} 1 & & & \\ & \cos\theta_2 & & -\sin\theta_2 \\ & & 1 & \\ & +\sin\theta_2 & & \cos\theta_2 \end{bmatrix} \approx I_4 + \begin{bmatrix} & & & \\ & & & -\theta_2 \\ & & & \\ & \theta_2 & & \end{bmatrix} \Rightarrow \omega^1{}_3 = -\omega^3{}_1 = -\theta_2$

$\omega^{13} = \theta_2$; $\omega^{31} = -\theta_2$

$U(\theta_2) = e^{-\frac{i}{2}(\omega^{13}M_{13} + \omega^{31}M_{31})} = e^{-i\omega^{31}M_{31}} = e^{+i\theta_2 M_{31}} \Rightarrow M_{31} = -i\left.\frac{dU(\theta_2)}{d\theta_2}\right|_{\theta_2=0}$

$M_{31} = \begin{bmatrix} & & & \\ & & & i \\ & & & \\ & -i & & \end{bmatrix}$

∘ око z осе за угао θ_3; $U(\theta_3) = \begin{bmatrix} 1 & & & \\ & \cos\theta_3 & \sin\theta_3 & \\ & -\sin\theta_3 & \cos\theta_3 & \\ & & & 1 \end{bmatrix} \approx I_4 + \begin{bmatrix} & & & \\ & & \theta_3 & \\ & -\theta_3 & & \\ & & & \end{bmatrix} \Rightarrow \omega^1{}_2 = -\omega^2{}_1 = \theta_3$

$\omega^{12} = -\theta_3$

$U(\theta_3) = e^{-\frac{i}{2}(\omega^{12}M_{12} + \omega^{21}M_{21})} = e^{+i\theta_3 M_{12}} \Rightarrow M_{12} = -i\left.\frac{dU(\theta_3)}{d\theta_3}\right|_{\theta_3=0}$

$M_{12} = \begin{bmatrix} & & & \\ & & -i & \\ & i & & \\ & & & \end{bmatrix}$

Бустови:

∘ дуж x-осе $U(\beta_1) = \begin{bmatrix} \operatorname{ch}\beta_1 & -\operatorname{sh}\beta_1 & & \\ -\operatorname{sh}\beta_1 & \operatorname{ch}\beta_1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ $\operatorname{th}\beta_1 = \frac{v_1}{c}$ $U(\beta_1) \approx I_4 + \begin{bmatrix} & -\beta_1 & & \\ -\beta_1 & & & \\ & & & \\ & & & \end{bmatrix} \Rightarrow \omega^0{}_1 = +\omega^1{}_0 = -\omega^{01} = -\beta_1$

$U(\beta_1) = e^{-\frac{i}{2}(\omega^{01}M_{01} + \omega^{10}M_{10})} = e^{-i\omega^{01}M_{01}} = e^{-i\beta_1 M_{01}} \Rightarrow M_{01} = i\left.\frac{dU(\beta_1)}{d\beta_1}\right|_{\beta_1=0} = \begin{bmatrix} & -i & & \\ -i & & & \\ & & & \\ & & & \end{bmatrix}$

∘ дуж y-осе $U(\beta_2) = \begin{bmatrix} \operatorname{ch}\beta_2 & & -\operatorname{sh}\beta_2 & \\ & 1 & & \\ -\operatorname{sh}\beta_2 & & \operatorname{ch}\beta_2 & \\ & & & 1 \end{bmatrix}$; $\operatorname{th}\beta_2 = \frac{v_2}{c}$ $U(\beta_2) = I_4 + \begin{bmatrix} & & -\beta_2 & \\ & & & \\ -\beta_2 & & & \\ & & & \end{bmatrix}$; $\omega^0{}_2 = \omega^2{}_0 = -\omega^{02} = -\beta_2$

$U(\beta_2) = e^{-\frac{i}{2}(\omega^{02}M_{02} + \omega^{20}M_{20})} = e^{-i\omega^{02}M_{02}} = e^{-i\beta_2 M_{02}}$, $M_{02} = i\left.\frac{dU(\beta_2)}{d\beta_2}\right|_{\beta_2=0} = \begin{bmatrix} & & -i & \\ & & & \\ -i & & & \\ & & & \end{bmatrix}$

∘ дуж z-осе $U(\beta_3) = \begin{bmatrix} \operatorname{ch}\beta_3 & & & -\operatorname{sh}\beta_3 \\ & 1 & & \\ & & 1 & \\ -\operatorname{sh}\beta_3 & & & \operatorname{ch}\beta_3 \end{bmatrix}$; $\operatorname{th}\beta_3 = \frac{v_3}{c}$ $U(\beta_3) = I_4 + \begin{bmatrix} & & & -\beta_3 \\ & & & \\ & & & \\ -\beta_3 & & & \end{bmatrix}$; $\omega^0{}_3 = \omega^3{}_0 = -\omega^{03} = -\beta_3$

$U(\beta_3) = e^{-\frac{i}{2}(\omega^{03}M_{03} + \omega^{30}M_{30})} = e^{-i\omega^{03}M_{03}} = e^{-i\beta_3 M_{03}}$, $M_{03} = i\left.\frac{dU(\beta_3)}{d\beta_3}\right|_{\beta_3=0} = \begin{bmatrix} & & & -i \\ & & & \\ & & & \\ -i & & & \end{bmatrix}$

1.4. Одредити матричне елементе генератора Лоренцове групе у дефиниционој репрезентацији.

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} = (\delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}) x^{\nu} \quad \text{за инфинитезималне Л.Т.}$$

$$= (U(\Lambda))^{\mu}{}_{\nu} x^{\nu} = \left(e^{\frac{i}{2}\omega_{\alpha\beta} M^{\alpha\beta}}\right)^{\mu}{}_{\nu} x^{\nu} \approx \left(I^{\mu}{}_{\nu} - \frac{i}{2}\omega^{\alpha\beta}(M_{\alpha\beta})^{\mu}{}_{\nu}\right) x^{\nu}$$

$$= x^{\mu} - \frac{i}{2}\omega^{\alpha\beta}(M_{\alpha\beta})^{\mu}{}_{\nu} x^{\nu}$$

$$\Rightarrow \quad \omega^{\mu}{}_{\nu} x^{\nu} = -\frac{i}{2}\omega^{\alpha\beta}(M_{\alpha\beta})^{\mu}{}_{\nu} x^{\nu}$$

$$\Rightarrow \quad \omega^{\mu}{}_{\nu} = -\frac{i}{2}\omega^{\alpha\beta}(M_{\alpha\beta})^{\mu}{}_{\nu} = \omega^{\alpha\beta}\,\delta^{\mu}_{\alpha} g_{\beta\nu} = \frac{1}{2}\omega^{\alpha\beta}(\delta^{\mu}_{\alpha} g_{\beta\nu} - \delta^{\mu}_{\beta} g_{\alpha\nu})$$

$$\Rightarrow \boxed{(M_{\alpha\beta})^{\mu}{}_{\nu} = i(\delta^{\mu}_{\alpha} g_{\beta\nu} - \delta^{\mu}_{\beta} g_{\alpha\nu})}$$

$$M_{01} = i\begin{bmatrix} & -1 & & \\ -1 & & & \\ & & & \\ & & & \end{bmatrix} \qquad M_{02} = i\begin{bmatrix} & & -1 & \\ & & & \\ -1 & & & \\ & & & \end{bmatrix} \qquad M_{03} = i\begin{bmatrix} & & & -1 \\ & & & \\ & & & \\ -1 & & & \end{bmatrix}$$

$$M_{12} = i\begin{bmatrix} & & & \\ & & -1 & \\ & 1 & & \\ & & & \end{bmatrix} \qquad M_{23} = i\begin{bmatrix} & & & \\ & & & \\ & & & -1 \\ & & 1 & \end{bmatrix} \qquad M_{31} = i\begin{bmatrix} & & & \\ & & & 1 \\ & & & \\ & -1 & & \end{bmatrix}$$

$$M_3 \qquad\qquad M_1 \qquad\qquad M_2$$

1.6' Одредити комутатор између генератора ротација и ротација, ротација и бустова као и бустова и ротација.

$N_i = M_{0i}$ - генератори бустова дуж правца i

$M_i = \frac{1}{2}\varepsilon_{ijk} M_{jk}$ - генератори ротација око правца i

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho})$$

Комутационе релације Лоренцове алгебре

• ротације и ротације

$$[M_i, M_j] = \frac{1}{4}\varepsilon_{ikl}\varepsilon_{jmn}[M_{kl}, M_{mn}] = \frac{i}{4}\varepsilon_{ikl}\varepsilon_{jmn}(g_{kn}M_{lm} + g_{lm}M_{kn} - g_{km}M_{ln} - g_{ln}M_{km})$$

$$= -\frac{i}{4}(\varepsilon_{ikl}\varepsilon_{jmk}M_{lm} + \varepsilon_{ikl}\varepsilon_{jln}M_{kn} - \varepsilon_{ikl}\varepsilon_{jkn}M_{ln} - \varepsilon_{ikl}\varepsilon_{jml}M_{km})$$

$$= -\frac{i}{4}(-(\delta_{ij}\cancelto{0}{\delta_{lm}} - \delta_{im}\delta_{lj})M_{lm} - (\delta_{ij}\cancelto{0}{\delta_{kn}} - \delta_{in}\delta_{kj})M_{kn} - (\delta_{ij}\cancelto{0}{\delta_{ln}} - \delta_{in}\delta_{lj})M_{ln} - (\delta_{ij}\cancelto{0}{\delta_{km}} - \delta_{im}\delta_{kj})M_{km})$$

$$= -iM_{ji} = iM_{ij} = iM_{nm}\varepsilon_{ijk}\varepsilon_{nmk}\frac{1}{2} = i\varepsilon_{ijk}M_k$$

• ротације и бустови

$$[M_i, N_j] = \frac{1}{2}\varepsilon_{ikl}[M_{kl}, M_{0j}] = \frac{i}{2}\varepsilon_{ikl}(g_{kj}M_{l0} + \cancelto{0}{g_{l0}}M_{kj} - \cancelto{0}{g_{k0}}M_{lj} - g_{lj}M_{k0})$$

$$= \frac{i}{2}(-\varepsilon_{ijl}M_{l0} + \varepsilon_{ikj}M_{k0}) = i\varepsilon_{ijk}M_{0k} = i\varepsilon_{ijk}N_k$$

∘ бустови и бустови

$$[N_i,N_j]=[M_{0i},M_{0j}]=i\left(\cancel{g_{0j}}^{\,0} M_{i0}+\cancel{g_{i0}}^{\,0} M_{0j}-g_{00}^{\,1} M_{ij}-g_{ij}\cancel{M_{00}}^{\,0}\right)=-iM_{ij}$$
$$=-i\varepsilon_{ijk}\,\varepsilon_{kmn}M_{mn}\frac{1}{2}=-i\varepsilon_{ijk}M_k$$

1.5. Показати да је дефинициона репрезентација Лоренцове групе $\left(\frac{1}{2},\frac{1}{2}\right)$ репрезентација.

$$M_i=\frac{1}{2}\varepsilon_{ijk}M_{jk},\quad N_i=M_{0i}\implies A_i=\frac{1}{2}(M_i+iN_i),\quad B_i=\frac{1}{2}(M_i-iN_i)$$

комплексификација $SO(1,3)_{\mathbb{R}}$ алгебре у $su(2)\otimes su(2)$

$$[A_i,A_j]=\frac{1}{4}[M_i+iN_i,M_j+iN_j]=\frac{1}{4}\left(i\varepsilon_{ijk}M_k+i\,i\varepsilon_{ijk}N_k-i\,i\varepsilon_{jik}N_k-(-i\varepsilon_{ijk}M_k)\right)$$
$$=\frac{1}{2}\left(i\varepsilon_{ijk}M_k-\varepsilon_{ijk}N_k\right)=i\varepsilon_{ijk}A_k\quad\rightarrow su(2) \text{ алгебра}$$

$$[B_i,B_j]=\frac{1}{4}[M_i-iN_i,M_j-iN_j]=\frac{1}{4}\left(i\varepsilon_{ijk}M_k-i\cdot i\varepsilon_{ijk}N_k+i\cdot i\varepsilon_{jik}N_k-(-i\varepsilon_{ijk}M_k)\right)$$
$$=\frac{1}{2}\left(i\varepsilon_{ijk}M_k+\varepsilon_{ijk}N_k\right)=i\varepsilon_{ijk}B_k\quad\rightarrow su(2) \text{ алгебра}$$

$$[A_i,B_j]=\frac{1}{4}[M_i+iN_i,M_j-iN_j]=\frac{1}{4}\left(i\varepsilon_{ijk}M_k-\cancel{i\cdot i\varepsilon_{ijk}N_k}-\cancel{i\,i\varepsilon_{jik}N_k}-i\varepsilon_{ijk}M_k\right)=0$$

добили смо две $su(2)$ алгебре, њихови Казимирови оператори карактеришу репрезентације.

дефинициона репрезентација $(M_{\alpha\beta})^{\mu}{}_{\nu}=i\left(\delta^{\mu}_{\alpha}g_{\beta\nu}-\delta^{\mu}_{\beta}g_{\alpha\nu}\right)$

$$M_1=M_{23},\ M_2=M_{31},\ M_3=M_{12},\ N_1=M_{01},\ N_2=M_{02},\ N_3=M_{03}$$
$$A_1=\frac{1}{2}(M_{23}+iM_{01}),\ A_2=\frac{1}{2}(M_{31}+iM_{02}),\ A_3=\frac{1}{2}(M_{12}+iM_{03})$$
$$B_1=\frac{1}{2}(M_{23}-iM_{01}),\ B_2=\frac{1}{2}(M_{31}-iM_{02}),\ B_3=\frac{1}{2}(M_{12}-iM_{03})$$

$$A_1^2=-\frac{1}{4}\begin{bmatrix} & -i & & \\ -i & & & \\ & & & -1 \\ & & 1 & \end{bmatrix}^2=\frac{1}{4}I;\quad A_2^2=-\frac{1}{4}\begin{bmatrix} & & -i & \\ & & & 1 \\ -i & & & \\ & -1 & & \end{bmatrix}^2=\frac{1}{4}I,\quad A_3^2=-\frac{1}{4}\begin{bmatrix} & & & -i \\ & & -1 & \\ & 1 & & \\ -i & & & \end{bmatrix}^2=\frac{1}{4}I$$

$$B_1^2=-\frac{1}{4}\begin{bmatrix} & i & & \\ i & & & \\ & & & -1 \\ & & 1 & \end{bmatrix}^2=\frac{1}{4}I;\quad B_2^2=-\frac{1}{4}\begin{bmatrix} & & i & \\ & & & 1 \\ i & & & \\ & -1 & & \end{bmatrix}^2=\frac{1}{4}I;\quad B_3^2=-\frac{1}{4}\begin{bmatrix} & & & i \\ & & -1 & \\ & 1 & & \\ i & & & \end{bmatrix}^2=\frac{1}{4}I$$

$$\implies \vec{A}^2=\frac{3}{4}I=j_1(j_1+1)I\implies j_1=\frac{1}{2}$$
$$\vec{B}^2=\frac{3}{4}I=j_2(j_2+1)I\implies j_2=\frac{1}{2}$$

деф. репрезентација је $\left(\frac{1}{2},\frac{1}{2}\right)$

Произвољна матрица $A \in SL(2,\mathbb{C})$ може да се напише као $A = UH$, где је:
$U = \cos\frac{\varphi}{2}\, I + i(\vec{\sigma}\vec{n})\sin\frac{\varphi}{2}$, $H = \cosh\frac{\psi}{2}\, I + \vec{\sigma}\vec{\eta}\,\sinh\frac{\psi}{2}$. Показати да U одговара ротацији за угао φ око осе $\vec{n}$, док H одговара бусту дуж правца $\vec{\eta}$ са брзином $\frac{v}{c} = \tanh\psi$.

Веза $L_+^\uparrow$ и $SL(2,\mathbb{C})$

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$; $\Lambda \in L_+^\uparrow$ $x^\mu \in M_4$ простор Минковског

Придружимо четворовектору $x = (x^0, x^1, x^2, x^3)^T$ хермитску матрицу X

$$X = \sigma_\mu x^\mu \;;\; \sigma^\mu = (I, \vec{\sigma}) \;;\; X = \begin{array}{|c|c|} \hline ct-z & -x+iy \\ \hline -x-iy & ct+z \\ \hline \end{array} \;;\; \bar{\sigma}^\mu \overset{def}{=} (1, -\vec{\sigma})$$

$$\det X = c^2t^2 - z^2 - x^2 - y^2 = x^2$$

Трансформације $A \in SL(2,\mathbb{C})$: $X \to X' = AXA^+$ не мењају детерминанту X

$$\det X' = \underbrace{\det A}_{1} \cdot \underbrace{\det A^+}_{1} \det X = \det X$$

A – комплексна 2x2 матрица са јединичном детерминантом

$$tr(X\bar{\sigma}^\mu) = tr(x_\nu \sigma^\nu \bar{\sigma}^\mu) = x_\nu 2g^{\mu\nu} = 2x^\mu \;;\; x^\mu = \frac{1}{2} tr(X\bar{\sigma}^\mu) \quad (tr\,\sigma^\mu\bar{\sigma}^\nu = 2g^{\mu\nu})$$

$$x'^\mu = \frac{1}{2} tr(X'\bar{\sigma}^\mu) = \frac{1}{2} tr(AXA^+\bar{\sigma}^\mu) = \frac{1}{2} tr(A\sigma_\nu A^+\bar{\sigma}^\mu)\, x^\nu = \Lambda^\mu{}_\nu x^\nu$$

$$\Rightarrow \Lambda^\mu{}_\nu = \frac{1}{2} tr(A\sigma_\nu A^+\bar{\sigma}^\mu) = \boxed{\frac{1}{2} tr(\bar{\sigma}^\mu A\sigma_\nu A^+)}$$

и A и $-A$ одговарају истој Λ па је $SL(2,\mathbb{C})$ универзално наткривајућа група од $L_+^\uparrow$

Ако напишемо A као $A = e^S$; $trS = 0$ јер $\det A = 1$ $\Rightarrow$ A има 6 $\mathbb{R}$ парам.

$S = S_1 + S_2$ $S_1 = +\frac{1}{2} i\varphi\vec{\sigma}\vec{n}$; $S_2 = +\frac{1}{2}\psi\vec{\sigma}\vec{\eta}$ ($\vec{n}$ и $\vec{\eta}$, φ, ψ ← 6 реалних парам.)
($\vec{\sigma}$ и $i\vec{\sigma}$ 6 независних 2x2 матр. са трагом 0)

$$U = e^{S_1} = \sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(+\frac{1}{2} i\varphi\vec{\sigma}\vec{n}\right)^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\left(+\frac{1}{2} i\varphi\vec{\sigma}\vec{n}\right)^{2k+1}$$

$(\vec{\sigma}\vec{n} \cdot \vec{\sigma}\vec{n} = \sigma_i\sigma_j n_i n_j = (\delta_{ij} + \overset{0}{\overbrace{i\varepsilon_{ijk}\sigma_k}})n_i n_j = 1)$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k\left(\frac{\varphi}{2}\right)^{2k} + i\vec{\sigma}\vec{n}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{\varphi}{2}\right)^{2k+1} = \cos\frac{\varphi}{2} + i\vec{\sigma}\vec{n}\sin\frac{\varphi}{2}$$

$$H = e^{S_2} = \sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(\frac{1}{2}\psi\vec{\sigma}\vec{\eta}\right)^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\left(\frac{1}{2}\psi\vec{\sigma}\vec{\eta}\right)^{2k+1} =$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(\frac{\psi}{2}\right)^{2k} + \vec{\sigma}\vec{\eta}\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\left(\frac{\psi}{2}\right)^{2k+1} = ch\frac{\psi}{2} + \vec{\sigma}\vec{\eta}\, sh\frac{\psi}{2}$$

$U^+ = \cos\frac{\varphi}{2} - i\vec{\sigma}\vec{n}\sin\frac{\varphi}{2} \Rightarrow UU^+ = 1$ $U \in SU(2)$; $H^+ = H$ – хермитска матрица

• за $\vec{n} = \vec{e}_z$ $\quad U = \cos\frac{\varphi}{2} + i\sigma_3 \sin\frac{\varphi}{2} = \begin{pmatrix} e^{i\varphi/2} & \\ & e^{-i\varphi/2} \end{pmatrix}$

$$X \to X' = \begin{pmatrix} ct'-z' & -x'+iy' \\ -x'-iy' & ct'+z' \end{pmatrix} = U X U^+ = \begin{pmatrix} e^{i\varphi/2} & \\ & e^{-i\varphi/2} \end{pmatrix} \begin{pmatrix} ct-z & -x+iy \\ -x-iy & ct+z \end{pmatrix} \begin{pmatrix} e^{-i\varphi/2} & \\ & e^{i\varphi/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\varphi/2} & \\ & e^{-i\varphi/2} \end{pmatrix} \begin{pmatrix} e^{-i\varphi/2}(ct-z) & e^{i\varphi/2}(-x+iy) \\ e^{-i\varphi/2}(-x-iy) & e^{i\varphi/2}(ct+z) \end{pmatrix} = \begin{pmatrix} ct-z & e^{i\varphi}(-x+iy) \\ e^{-i\varphi}(-x-iy) & ct+z \end{pmatrix}$$

$\Rightarrow t' = t,\ z' = z\ ;\ x' = x\cos\varphi + y\sin\varphi\ ;\ y' = -x\sin\varphi + y\cos\varphi$

Ово је ротација око z-осе

• за $\vec{n} = \vec{e}_z$ $\quad H = ch\frac{\psi}{2} + \sigma_3 sh\frac{\psi}{2} = \begin{pmatrix} e^{\psi/2} & \\ & e^{-\psi/2} \end{pmatrix}$

$$X \to X' = \begin{pmatrix} ct'-z' & -x'+iy' \\ -x'-iy' & ct'+z' \end{pmatrix} = HXH^+ = HXH = \begin{pmatrix} e^{\psi/2} & \\ & e^{-\psi/2} \end{pmatrix} \begin{pmatrix} ct-z & -x+iy \\ -x-iy & ct+z \end{pmatrix} \begin{pmatrix} e^{\psi/2} & \\ & e^{-\psi/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\psi/2} & \\ & e^{-\psi/2} \end{pmatrix} \begin{pmatrix} e^{\psi/2}(ct-z) & e^{-\psi/2}(-x+iy) \\ e^{\psi/2}(-x-iy) & e^{-\psi/2}(ct+z) \end{pmatrix} = \begin{pmatrix} e^{\psi}(ct-z) & -x+iy \\ -x-iy & e^{-\psi}(ct+z) \end{pmatrix}$$

$\Rightarrow x' = x,\ y' = y \quad ct' = \frac{1}{2}e^{\psi}(ct-z) + \frac{1}{2}e^{-\psi}(ct+z) = ct\cdot ch\psi - z\, sh\psi;\ z' = -ct\, sh\psi + z\, ch\psi$

Ово је буст дуж z-осе

$th\psi = \frac{v}{c} = \frac{sh\psi}{ch\psi} \qquad ch^2\psi - sh^2\psi = 1 \Rightarrow sh\psi = \sqrt{ch^2\psi - 1}$

$\frac{v}{c} = \frac{\sqrt{ch^2\psi-1}}{ch\psi} \qquad \frac{v^2}{c^2}ch^2\psi = ch^2\psi - 1 \Rightarrow ch\psi = \frac{1}{\sqrt{1-v^2/c^2}}\ ;\ sh\psi = \frac{v/c}{\sqrt{1-v^2/c^2}}$

Поенкареова група

Група трансформација у простору Минковског које чувају растојање међу тачкама
Састоји се из Лоренцових трансформација и транслација.

$P = T^4 \wedge O(1,3,\mathbb{R})$ $(\Lambda, a) \in P$, $\Lambda \in O(1,3,\mathbb{R})$ a - четворовектор

$$(\Lambda, a)\, x^\mu = x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

Лоренцове трансформације су подгрупа Поенкареове групе, а транслације су инваријантна подгрупа Абелова.

Група је 10-параметарска.

$$U(\Lambda, a) = e^{-\frac{i}{2}\omega^{\mu\nu} M_{\mu\nu} + i a^\mu P_\mu}$$ репрезентација Поенкареове групе

$M_{\mu\nu}$ - генератори Лоренцове групе - ротације и бустови

P_μ - генератори транслација - енергија и импулс

Дата је трансформационо правило под дејством Поенкареове групе: $(\Lambda,a)x=\Lambda x+a$. Наћи закон множења, јединични и инверзни елемент.

$$(\Lambda_1,a_1)(\Lambda_2,a_2)x=(\Lambda_1,a_1)(\Lambda_2x+a_2)=\Lambda_1(\Lambda_2x+a_2)+a_1=\Lambda_1\Lambda_2x+\Lambda_1a_2+a_1$$
$$=(\Lambda_1\Lambda_2,\Lambda_1a_2+a_1)x$$
$$\Rightarrow\boxed{(\Lambda_1,a_1)(\Lambda_2,a_2)=(\Lambda_1\Lambda_2,\Lambda_1a_2+a_1)}$$

јединични елемент:

$$(\Lambda,a)(J,j)=(J,j)(\Lambda,a)=(\Lambda,a)$$
$$(\Lambda J;\Lambda j+a)=(J\Lambda,Ja+j)=(\Lambda,a)$$
$$\Rightarrow\ \Lambda J=J\Lambda=\Lambda\ \Rightarrow\boxed{J=I}\qquad \Lambda j+a=Ja+j=a\ \Rightarrow\boxed{j=0}$$

јединични елемент је $(I,0)$

инверз:

$$(\Lambda,a)(\Lambda,a)^{-1}=(I,0)\qquad(\Lambda,a)^{-1}=(\Lambda',a')$$
$$(\Lambda,a)(\Lambda',a')=(\Lambda\Lambda',\Lambda a'+a)=(I,0)\Rightarrow\ \Lambda'=\Lambda^{-1};\ a'=-\Lambda^{-1}a$$
$$\boxed{(\Lambda,a)^{-1}=(\Lambda^{-1},-\Lambda^{-1}a)}$$

Проверити закон множења у Поенкареовој групи $U^{-1}(\Lambda,0)U(1,\varepsilon)U(\Lambda,0)=U(1,\Lambda^{-1}\varepsilon)$, затим показати да из ове релације следи: $U^{-1}(\Lambda,0)P_\mu U(\Lambda,0)=(\Lambda^{-1})^\nu{}_\mu P_\nu$. Израчунати комутатор $[M_{\mu\nu},P_\rho]$

$$U^{-1}(\Lambda,0)U(1,\varepsilon)U(\Lambda,0)=U(\Lambda^{-1},0)U(1,\varepsilon)U(\Lambda,0)=U(\Lambda^{-1},\Lambda^{-1}\varepsilon)U(\Lambda,0)=U(1,\Lambda^{-1}\varepsilon)$$
$$U(1,\varepsilon)=e^{iP_\mu\varepsilon^\mu}\qquad\boxed{U(\Lambda,\varepsilon)=e^{-\frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}+iP_\mu\varepsilon^\mu}}$$

за мало ε: $U(1,\varepsilon)\simeq 1+iP_\mu\varepsilon^\mu$

$$U(1,\Lambda^{-1}\varepsilon)\approx 1+iP_\nu(\Lambda^{-1}\varepsilon)^\nu=1+iP_\nu(\Lambda^{-1})^\nu{}_\mu\varepsilon^\mu$$
$$U^{-1}(\Lambda,0)(1+iP_\mu\varepsilon^\mu)U(\Lambda,0)=1+iP_\nu(\Lambda^{-1})^\nu{}_\mu\varepsilon^\mu$$
$$\cancel{1}+\boxed{i\varepsilon^\mu}U^{-1}(\Lambda,0)P_\mu U(\Lambda,0)=\cancel{1}+\boxed{i\varepsilon^\mu}P_\nu(\Lambda^{-1})^\nu{}_\mu$$

за произвољно ε^μ важи $\Rightarrow$

$$\boxed{U^{-1}(\Lambda,0)P_\mu U(\Lambda,0)=P_\nu(\Lambda^{-1})^\nu{}_\mu}$$

сада развијемо и $U(\Lambda,0)$ за $\Lambda^\mu{}_\nu\approx\delta^\mu{}_\nu+\omega^\mu{}_\nu$ где је ω мало!

$$e^{\frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}}P_\rho\, e^{-\frac{i}{2}M_{\mu'\nu'}\omega^{\mu'\nu'}}=(\Lambda^{-1})^\sigma{}_\rho P_\sigma$$
$$\left(1+\frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}\right)P_\rho\left(1-\frac{i}{2}M_{\mu'\nu'}\omega^{\mu'\nu'}\right)\approx(\delta^\sigma_\rho-\omega^\sigma{}_\rho)P_\sigma=P_\rho-\omega^{\mu\nu}P_\mu g_{\nu\rho}$$

$$\cancel{P_\rho} + \frac{i}{2}\omega^{\mu\nu}[M_{\mu\nu}, P_\rho] = \cancel{P_\rho} - \omega^{\mu\nu} \underbrace{g_{\nu\rho}P_\mu}_{\text{антисиметризујемо по } \mu \text{ и } \nu}$$

$$\frac{i}{2}\omega^{\mu\nu}[M_{\mu\nu}, P_\rho] = -\omega^{\mu\nu}\frac{1}{2}(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu)$$ сада можемо да скратимо $\omega^{\mu\nu}$

$$\boxed{[M_{\mu\nu}, P_\rho] = i\,(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu)}$$

Показати да важи $U^{-1}(\Lambda,0)U(\Lambda',0)U(\Lambda,0) = U(\Lambda^{-1}\Lambda'\Lambda, 0)$, па израчунати $[M_{\mu\nu}, M_{\rho\sigma}]$

$$U^{-1}(\Lambda,0)U(\Lambda',0)U(\Lambda,0) = U(\Lambda^{-1},0)U(\Lambda',0)U(\Lambda,0) = U(\Lambda^{-1}\Lambda',0)U(\Lambda,0) = U(\Lambda^{-1}\Lambda'\Lambda,0)$$

развијемо $U(\Lambda',0) = e^{-\frac{i}{2}M_{\mu\nu}\omega'^{\mu\nu}} \approx \mathbb{1} - \frac{i}{2}M_{\mu\nu}\omega'^{\mu\nu}$; $\Lambda' = \delta + \omega'$

$$U(\Lambda^{-1}\Lambda'\Lambda, 0) = e^{-\frac{i}{2}M_{\mu\nu}\omega''^{\mu\nu}} \approx \mathbb{1} - \frac{i}{2}M_{\mu\nu}\omega''^{\mu\nu} \quad ; \Lambda^{-1}\Lambda'\Lambda = \delta + \omega''$$

$$(\Lambda^{-1})^\mu{}_\nu \Lambda'^\nu{}_\rho \Lambda^\rho{}_\sigma = \delta^\mu{}_\sigma + \omega''^\mu{}_\sigma$$

$$(\Lambda^{-1})^\mu{}_\nu (\delta^\nu{}_\rho + \omega'^\nu{}_\rho)\Lambda^\rho{}_\sigma = \delta^\mu{}_\sigma + \omega''^\mu{}_\sigma$$

$$\cancel{\delta^\mu{}_\sigma} + (\Lambda^{-1})^\mu{}_\nu \omega'^\nu{}_\rho \Lambda^\rho{}_\sigma = \cancel{\delta^\mu{}_\sigma} + \omega''^\mu{}_\sigma$$

$$\Lambda^T g \Lambda = g$$

$$\boxed{g^{-1}\Lambda^T g = \Lambda^{-1}}$$

$$(\Lambda^{-1})^\mu{}_\nu = g^{\mu\alpha}(\Lambda^T)_\alpha{}^\beta g_{\beta\nu} = g^{\mu\alpha}\Lambda^\beta{}_\alpha g_{\beta\nu} = \Lambda_\nu{}^\mu$$

$$\Rightarrow \omega''^{\mu\sigma} = (\Lambda^{-1})^\mu{}_\nu \omega'^\nu{}_\rho \Lambda^\rho{}_\tau g^{\tau\sigma} = (\Lambda^{-1})^\mu{}_\nu \omega'^{\nu\lambda}\boxed{g_{\lambda\rho}\Lambda^\rho{}_\tau g^{\tau\sigma}}$$

$$= (\Lambda^{-1})^\mu{}_\nu \omega'^{\nu\lambda}\Lambda_\lambda{}^\sigma = (\Lambda^{-1})^\mu{}_\nu (\Lambda^{-1})^\sigma{}_\lambda \omega'^{\nu\lambda}$$

$$\boxed{\omega''^{\mu\nu} = (\Lambda^{-1})^\mu{}_\rho (\Lambda^{-1})^\nu{}_\sigma \omega'^{\rho\sigma}}$$ ово вратимо у закон множења!

$$U^{-1}(\Lambda,0)\left(\mathbb{1} - \frac{i}{2}M_{\mu\nu}\omega'^{\mu\nu}\right)U(\Lambda,0) = \mathbb{1} - \frac{i}{2}M_{\rho\sigma}\omega''^{\rho\sigma} = \mathbb{1} - \frac{i}{2}M_{\rho\sigma}(\Lambda^{-1})^\rho{}_\mu(\Lambda^{-1})^\sigma{}_\nu\omega'^{\mu\nu}$$

$$\cancel{\mathbb{1}} - \frac{i}{2}\omega'^{\mu\nu}U^{-1}(\Lambda,0)M_{\mu\nu}U(\Lambda,0) = \cancel{\mathbb{1}} - \frac{i}{2}\omega'^{\mu\nu}\underbrace{(\Lambda^{-1})^\rho{}_\mu(\Lambda^{-1})^\sigma{}_\nu M_{\rho\sigma}}_{\text{антисиметризујемо по } \mu,\nu}$$

$$\omega'^{\mu\nu}U^{-1}(\Lambda,0)M_{\mu\nu}U(\Lambda,0) = \omega'^{\mu\nu}\frac{1}{2}\left[(\Lambda^{-1})^\rho{}_\mu(\Lambda^{-1})^\sigma{}_\nu - (\Lambda^{-1})^\rho{}_\nu(\Lambda^{-1})^\sigma{}_\mu\right]M_{\rho\sigma}$$ (нема потребе због $M_{\rho\sigma}$)

$$\Rightarrow \boxed{U^{-1}(\Lambda,0)M_{\mu\nu}U(\Lambda,0) = \frac{1}{2}\left[(\Lambda^{-1})^\rho{}_\mu(\Lambda^{-1})^\sigma{}_\nu - (\Lambda^{-1})^\rho{}_\nu(\Lambda^{-1})^\sigma{}_\mu\right]M_{\rho\sigma}} = \Lambda^{-1\,\rho}{}_\mu\Lambda^{-1\,\sigma}{}_\nu M_{\rho\sigma}$$

сада развијемо $U(\Lambda,0) = e^{-\frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}}$ $\quad (\Lambda^{-1})^\mu{}_\nu \approx \delta^\mu{}_\nu - \omega^\mu{}_\nu$

$$\left(\mathbb{1} + \frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}\right)M_{\rho\sigma}\left(\mathbb{1} - \frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}\right) = \frac{1}{2}\left((\delta^\mu{}_\rho - \omega^\mu{}_\rho)(\delta^\nu{}_\sigma - \omega^\nu{}_\sigma) - (\delta^\mu{}_\sigma - \omega^\mu{}_\sigma)(\delta^\nu{}_\rho - \omega^\nu{}_\rho)\right)M_{\mu\nu}$$

$$\cancel{M_{\rho\sigma}} + \frac{i}{2}\omega^{\mu\nu}[M_{\mu\nu}, M_{\rho\sigma}] = \frac{1}{2}\left(\cancel{M_{\rho\sigma}} - \cancel{M_{\sigma\rho}} - \omega^\mu{}_\rho M_{\mu\sigma} - M_{\rho\nu}\omega^\nu{}_\sigma + \omega^\mu{}_\sigma M_{\mu\rho} + \omega^\nu{}_\rho M_{\sigma\nu}\right)$$

$$\frac{i}{2}\omega^{\mu\nu}[M_{\mu\nu}, M_{\rho\sigma}] = \frac{1}{2}\omega^{\mu\nu}\underbrace{(-g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\rho\nu} + g_{\nu\sigma}M_{\mu\rho} + g_{\mu\rho}M_{\sigma\nu})}_{\text{ово је већ антисим. на } \mu\leftrightarrow\nu}$$

$$\Rightarrow \boxed{[M_{\mu\nu}, M_{\rho\sigma}] = i\,(g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho})}$$

Показати да важи $[P_\mu, P_\nu]=0$

$$U(\mathbb{1},\varepsilon)^{-1}U(\mathbb{1},\varepsilon')U(\mathbb{1},\varepsilon) = U(\mathbb{1},\varepsilon'-\varepsilon)U(\mathbb{1},\varepsilon) = U(\mathbb{1},\cancel{\varepsilon}+\varepsilon'-\cancel{\varepsilon}) = U(\mathbb{1},\varepsilon')$$

$$U(\mathbb{1},\varepsilon') = e^{iP_\nu\varepsilon'^\nu} \approx 1+iP_\nu\varepsilon'^\nu$$

$$U(\mathbb{1},\varepsilon)^{-1}(\mathbb{1}+iP_\nu\varepsilon'^\nu)U(\mathbb{1},\varepsilon) = \mathbb{1}+iP_\nu\varepsilon'^\nu$$

$$\cancel{\mathbb{1}} + i\varepsilon'^\nu U(\mathbb{1},\varepsilon)^{-1}P_\nu U(\mathbb{1},\varepsilon) = \cancel{\mathbb{1}}+iP_\nu\varepsilon'^\nu \Rightarrow \boxed{U^{-1}(\mathbb{1},\varepsilon)P_\nu U(\mathbb{1},\varepsilon) = P_\nu}$$

$$U(\mathbb{1},\varepsilon) = e^{iP_\mu\varepsilon^\mu} \approx 1+iP_\mu\varepsilon^\mu$$

$$(\mathbb{1}-iP_\mu\varepsilon^\mu)P_\nu(\mathbb{1}+iP_\mu\varepsilon^\mu) = P_\nu$$

$$\cancel{P_\nu} - i\varepsilon^\mu[P_\mu,P_\nu] = \cancel{P_\nu} \Rightarrow \boxed{[P_\mu,P_\nu]=0}$$

Вектор Паули-Лубанског дефинисан је као $W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma$.

а) Показати да важи: $W_\mu P^\mu = 0$ и $[W_\mu, P_\nu]=0$

$$W_\mu P^\mu = \frac{1}{2}\underbrace{\varepsilon_{\mu\nu\rho\sigma}}_{\text{асим. μσ}}M^{\nu\rho}\underbrace{P^\sigma P^\mu}_{\text{сим. μσ}} = 0 \quad \text{јер } [P^\mu,P^\nu]=0$$

$$[W_\mu,P_\nu] = [\tfrac{1}{2}\varepsilon_{\mu\rho\sigma\tau}M^{\rho\sigma}P^\tau, P_\nu] = \tfrac{1}{2}\varepsilon_{\mu\rho\sigma\tau}[M^{\rho\sigma},P_\nu]P^\tau$$

$$= \tfrac{1}{2}\varepsilon_{\mu\rho\sigma\tau}\, i(\delta^\sigma{}_\nu P^\rho - \delta^\rho{}_\nu P^\sigma)P^\tau = \tfrac{i}{2}\underbrace{\varepsilon_{\mu\rho\nu\tau}}_{\text{асим } \rho\tau}\underbrace{P^\rho P^\tau}_{\text{сим } \rho\tau} - \tfrac{i}{2}\underbrace{\varepsilon_{\mu\nu\sigma\tau}}_{\text{асим } \sigma\tau}\underbrace{P^\sigma P^\tau}_{\text{сим } \sigma\tau} = 0$$

б) Показати да је $W^2 = -\frac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 + M_{\mu\sigma}M^{\nu\sigma}P^\mu P_\nu$

$$W^2 = W_\mu W^\mu = \frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma\varepsilon^{\mu\alpha\beta\gamma}M_{\alpha\beta}P_\gamma$$

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta} = -\begin{vmatrix} \delta^\mu{}_\alpha & \delta^\mu{}_\beta & \delta^\mu{}_\gamma & \delta^\mu{}_\delta \\ \delta^\nu{}_\alpha & \delta^\nu{}_\beta & \delta^\nu{}_\gamma & \delta^\nu{}_\delta \\ \delta^\rho{}_\alpha & \delta^\rho{}_\beta & \delta^\rho{}_\gamma & \delta^\rho{}_\delta \\ \delta^\sigma{}_\alpha & \delta^\sigma{}_\beta & \delta^\sigma{}_\gamma & \delta^\sigma{}_\delta \end{vmatrix}; \quad \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\alpha\beta\gamma} = -\begin{vmatrix} \delta^\nu{}_\alpha & \delta^\nu{}_\beta & \delta^\nu{}_\gamma \\ \delta^\rho{}_\alpha & \delta^\rho{}_\beta & \delta^\rho{}_\gamma \\ \delta^\sigma{}_\alpha & \delta^\sigma{}_\beta & \delta^\sigma{}_\gamma \end{vmatrix}$$

$$= -\delta^\nu{}_\alpha\delta^\rho{}_\beta\delta^\sigma{}_\gamma - \delta^\nu{}_\beta\delta^\rho{}_\gamma\delta^\sigma{}_\alpha - \delta^\nu{}_\gamma\delta^\rho{}_\alpha\delta^\sigma{}_\beta + \delta^\nu{}_\gamma\delta^\rho{}_\beta\delta^\sigma{}_\alpha + \delta^\nu{}_\alpha\delta^\rho{}_\gamma\delta^\sigma{}_\beta + \delta^\nu{}_\beta\delta^\rho{}_\alpha\delta^\sigma{}_\gamma$$

$$W^2 = \frac{1}{4}(-\delta^\alpha_\nu\delta^\beta_\rho\delta^\gamma_\sigma - \delta^\alpha_\sigma\delta^\beta_\nu\delta^\gamma_\rho - \delta^\alpha_\rho\delta^\beta_\sigma\delta^\gamma_\nu + \delta^\alpha_\nu\delta^\beta_\sigma\delta^\gamma_\rho + \delta^\alpha_\rho\delta^\beta_\nu\delta^\gamma_\sigma + \delta^\alpha_\sigma\delta^\beta_\rho\delta^\gamma_\nu)M^{\nu\rho}P^\sigma M_{\alpha\beta}P_\gamma$$

$$= \frac{1}{4}M^{\nu\rho}P^\sigma(-M_{\nu\rho}P_\sigma - M_{\sigma\nu}P_\rho - M_{\rho\sigma}P_\nu + M_{\nu\sigma}P_\rho + M_{\rho\nu}P_\sigma + M_{\sigma\rho}P_\nu)$$

$$= -\frac{1}{2}M^{\nu\rho}P^\sigma(M_{\nu\rho}P_\sigma + M_{\sigma\nu}P_\rho + M_{\rho\sigma}P_\nu) = -\frac{1}{2}M^{\nu\rho}\underbrace{P^\sigma M_{\nu\rho}}P_\sigma - \frac{1}{2}M^{\nu\rho}P^\sigma M_{\sigma\nu}P_\rho - \frac{1}{2}M^{\nu\rho}P^\sigma M_{\rho\sigma}P_\nu$$

$$= -\frac{1}{2}M^{\nu\rho}(M_{\nu\rho}P^\sigma + [P^\sigma, M_{\nu\rho}])P_\sigma - M^{\nu\rho}(M_{\sigma\nu}P^\sigma + [P^\sigma, M_{\sigma\nu}])P_\rho$$

$$= -\frac{1}{2}M^{\nu\rho}M_{\nu\rho}P^2 + \frac{1}{2}iM^{\nu\rho}\underbrace{(\delta^\sigma{}_\rho P_\nu - \delta^\sigma{}_\nu P_\rho)P_\sigma}_{0} - \underbrace{M^{\nu\rho}M_{\sigma\nu}P^\sigma P_\rho}_{\text{ком.}} + iM^{\nu\rho}\underbrace{(\delta^\sigma_\nu P_\sigma - \delta^\sigma_\sigma P_\nu)}_{-3P_\nu}P_\rho$$

$$= -\frac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 + M_{\mu\sigma}M^{\mu\rho}P^\sigma P_\rho.$$

в) Користећи се претходним резултатом, показати да W^2 и P^2 комутирају са генераторима Поенкареове групе.

$$[P^2, P_\mu] = 0$$

$$[P^2, M_{\mu\nu}] = [P^\rho P_\rho, M_{\mu\nu}] = [P^\rho, M_{\mu\nu}]P_\rho + P^\rho[P_\rho, M_{\mu\nu}]$$

$$= -i(\delta^\rho_\nu P_\mu - \delta^\rho_\mu P_\nu)P_\rho - iP^\rho(g_{\rho\nu}P_\mu - g_{\rho\mu}P_\nu) = -iP_\mu P_\nu + iP_\nu P_\mu - iP_\nu P_\mu + iP_\mu P_\nu = 0$$

$$[W^2, P_\mu] = [W^\nu W_\nu, P_\mu] = [W^\nu, P_\mu]W_\nu + W^\nu[W_\nu, P_\mu] = 0$$

$$[W^2, M_{\rho\sigma}] = [-\tfrac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 + M_{\mu\lambda}M^{\nu\lambda}P^\mu P_\nu, M_{\rho\sigma}]$$

$$= -\tfrac{1}{2}[M_{\mu\nu}, M_{\rho\sigma}]M^{\mu\nu}P^2 - \tfrac{1}{2}M_{\mu\nu}[M^{\mu\nu}, M_{\rho\sigma}]P^2 - \tfrac{1}{2}M_{\mu\nu}M^{\mu\nu}\overset{0}{[P^2, M_{\rho\sigma}]}$$

$$+ [M_{\mu\lambda}, M_{\rho\sigma}]M^{\nu\lambda}P^\mu P_\nu + M_{\mu\lambda}[M^{\nu\lambda}, M_{\rho\sigma}]P^\mu P_\nu + M_{\mu\lambda}M^{\nu\lambda}[P^\mu, M_{\rho\sigma}]P_\nu + M_{\mu\lambda}M^{\nu\lambda}P^\mu[P_\nu, M_{\rho\sigma}]$$

$$= -\tfrac{1}{2}i(g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho})M^{\mu\nu}P^2 - \tfrac{i}{2}M^{\mu\nu}(g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho})P^2$$

$$+ i(g_{\mu\sigma}M_{\lambda\rho} + g_{\lambda\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\lambda\sigma} - g_{\lambda\sigma}M_{\mu\rho})M^{\nu\lambda}P^\mu P_\nu + iM_{\mu\lambda}(\delta^\nu_\sigma M^\lambda{}_\rho + \delta^\lambda_\rho M^\nu{}_\sigma - \delta^\nu_\rho M^\lambda{}_\sigma - \delta^\lambda_\sigma M^\nu{}_\rho)P^\mu P_\nu$$

$$- iM_{\mu\lambda}M^{\nu\lambda}(\delta^\mu_\sigma P_\rho - \delta^\mu_\rho P_\sigma)P_\nu - iM_{\mu\lambda}M^{\nu\lambda}P^\mu(g_{\nu\sigma}P_\rho - g_{\nu\rho}P_\sigma)$$

$$= -\tfrac{i}{2}(M_{\nu\rho}M_\sigma{}^\nu + M_{\mu\sigma}M^\mu{}_\rho - M_{\nu\sigma}M_\rho{}^\nu - M_{\mu\rho}M^\mu{}_\sigma)P^2 - \tfrac{i}{2}(M_\sigma{}^\nu M_{\nu\rho} + M^\mu{}_\rho M_{\mu\sigma} - M_\rho{}^\nu M_{\nu\sigma} - M^\mu{}_\sigma M_{\mu\rho})P^2$$

$$+ i(M_{\lambda\rho}M^{\nu\lambda}P_\sigma P_\nu + M_{\mu\sigma}M^\nu{}_\rho P^\mu P_\nu - M_{\lambda\sigma}M^{\nu\lambda}P_\rho P_\nu - M_{\mu\rho}M^\nu{}_\sigma P^\mu P_\nu + M_{\mu\lambda}M^\lambda{}_\rho P^\mu P_\sigma + M_{\mu\rho}M^\nu{}_\sigma P^\mu P_\nu$$

$$- M_{\mu\lambda}M^\lambda{}_\sigma P^\mu P_\rho - M_{\mu\sigma}M^\nu{}_\rho P^\mu P_\nu - M_{\sigma\lambda}M^{\nu\lambda}P_\rho P_\nu + M_{\rho\lambda}M^{\nu\lambda}P_\sigma P_\nu - M_{\mu\lambda}M_\sigma{}^\lambda P^\mu P_\rho + M_{\mu\lambda}M_\rho{}^\lambda P^\mu P_\sigma)$$

$$= 0$$

W^2 и P^2 су Казимирови оператори Поенкареове алгебре.

Показати да важи:

a) $[M_{\mu\nu}, W_\sigma] = i(g_{\nu\sigma}W_\mu - g_{\mu\sigma}W_\nu)$

$$[M_{\mu\nu}, W_\sigma] = \frac{1}{2}\varepsilon_{\sigma\alpha\beta\gamma}[M_{\mu\nu}, M^{\alpha\beta}p^\gamma] = \frac{1}{2}\varepsilon_{\sigma\alpha\beta\gamma}\left([M_{\mu\nu}, M^{\alpha\beta}]p^\gamma + M^{\alpha\beta}[M_{\mu\nu}, p^\gamma]\right)$$

$$= \frac{i}{2}\varepsilon_{\sigma\alpha\beta\gamma}\left[(\delta_\mu^\beta M_\nu{}^\alpha + \delta_\nu^\alpha M_\mu{}^\beta - \delta_\mu^\alpha M_\nu{}^\beta - \delta_\nu^\beta M_\mu{}^\alpha)p^\gamma + M^{\alpha\beta}(\delta_\nu^\gamma p_\mu - \delta_\mu^\gamma p_\nu)\right]$$

$$= \frac{i}{2}\left(\underline{\varepsilon_{\sigma\alpha\mu\gamma}M_\nu{}^\alpha p^\gamma} + \underline{\underline{\varepsilon_{\sigma\nu\beta\gamma}M_\mu{}^\beta p^\gamma}} - \underline{\varepsilon_{\sigma\mu\beta\gamma}M_\nu{}^\beta p^\gamma} - \underline{\underline{\varepsilon_{\sigma\alpha\nu\gamma}M_\mu{}^\alpha p^\gamma}} + \varepsilon_{\sigma\alpha\beta\nu}M^{\alpha\beta}p_\mu - \varepsilon_{\sigma\alpha\beta\mu}M^{\alpha\beta}p_\nu\right)$$

$$= i\left(\varepsilon_{\sigma\alpha\mu\gamma}M_\nu{}^\alpha p^\gamma + \varepsilon_{\sigma\nu\beta\gamma}M_\mu{}^\beta p^\gamma + \frac{1}{2}\varepsilon_{\sigma\alpha\beta\nu}M^{\alpha\beta}p_\mu - \frac{1}{2}\varepsilon_{\sigma\alpha\beta\mu}M^{\alpha\beta}p_\nu\right.$$

$$= i\left(\varepsilon_{\sigma\alpha\mu\gamma}g_{\nu\beta}M^{\beta\alpha}p^\gamma + \varepsilon_{\sigma\nu\beta\gamma}g_{\mu\alpha}M^{\alpha\beta}p^\gamma + \frac{1}{2}\varepsilon_{\sigma\alpha\beta\nu}g_{\mu\gamma}M^{\alpha\beta}p^\gamma - \frac{1}{2}\varepsilon_{\sigma\alpha\beta\mu}g_{\nu\gamma}M^{\alpha\beta}p^\gamma\right)$$

$$= \frac{i}{2}\left(-\underline{\varepsilon_{\sigma\alpha\mu\gamma}g_{\nu\beta}} + \underline{\varepsilon_{\sigma\beta\mu\gamma}g_{\nu\alpha}} + \underline{\underline{\varepsilon_{\sigma\nu\beta\gamma}g_{\mu\alpha}}} - \underline{\underline{\varepsilon_{\sigma\nu\alpha\gamma}g_{\mu\beta}}} + \underline{\underline{\varepsilon_{\sigma\alpha\beta\nu}g_{\mu\gamma}}} - \underline{\varepsilon_{\sigma\alpha\beta\mu}g_{\nu\gamma}}\right)M^{\alpha\beta}p^\gamma$$

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$$\boxed{\varepsilon_{\mu\nu\rho\sigma}g_{\tau\lambda} - \varepsilon_{\mu\nu\rho\tau}g_{\sigma\lambda} + \varepsilon_{\mu\nu\sigma\tau}g_{\rho\lambda} - \varepsilon_{\mu\rho\sigma\tau}g_{\nu\lambda} + \varepsilon_{\nu\rho\sigma\tau}g_{\mu\lambda} = 0}$$

$$[M_{\mu\nu}, W_\sigma] = \frac{i}{2}\left(-\varepsilon_{\sigma\alpha\mu\gamma}g_{\beta\nu} + \varepsilon_{\sigma\mu\gamma\beta}g_{\alpha\nu} + \varepsilon_{\sigma\alpha\mu\beta}g_{\gamma\nu}\right.$$
$$\left. + \varepsilon_{\sigma\nu\beta\gamma}g_{\alpha\mu} + \varepsilon_{\sigma\nu\gamma\alpha}g_{\beta\mu} - \varepsilon_{\sigma\nu\beta\alpha}g_{\gamma\mu}\right)M^{\alpha\beta}p^\gamma$$

$$= \frac{i}{2}\left(\cancel{\varepsilon_{\sigma\alpha\gamma\beta}g_{\mu\nu}} + \varepsilon_{\alpha\mu\gamma\beta}g_{\sigma\nu} - \varepsilon_{\nu\beta\gamma\alpha}g_{\sigma\mu} + \cancel{\varepsilon_{\sigma\beta\gamma\alpha}g_{\nu\mu}}\right)M^{\alpha\beta}p^\gamma$$

$$= g_{\sigma\nu}\frac{i}{2}\varepsilon_{\mu\alpha\beta\gamma}M^{\alpha\beta}p^\gamma - g_{\sigma\mu}\frac{i}{2}\varepsilon_{\nu\alpha\beta\gamma}M^{\alpha\beta}p^\gamma = g_{\sigma\nu}W_\mu - g_{\sigma\mu}W_\nu$$

б) $[W_\mu, W_\nu] = -i\varepsilon_{\mu\nu\sigma\rho}W^\sigma p^\rho$

$$[W_\mu, W_\nu] = \frac{1}{2}\varepsilon_{\mu\alpha\beta\gamma}[M^{\alpha\beta}p^\gamma, W_\nu] = \frac{1}{2}\varepsilon_{\mu\alpha\beta\gamma}\left([M^{\alpha\beta}, W_\nu]p^\gamma + M^{\alpha\beta}\overset{0}{\cancel{[p^\gamma, W_\nu]}}\right)$$

$$= \frac{1}{2}\varepsilon_{\mu\alpha\beta\gamma}\, i(\delta_\nu^\beta W^\alpha - \delta_\nu^\alpha W^\beta)p^\gamma = \cancel{\frac{1}{2}}\, i\cdot\varepsilon_{\mu\alpha\beta\gamma}\cdot\cancel{2}\,\delta_\nu^\beta W^\alpha p^\gamma$$

$$= -i\varepsilon_{\mu\nu\rho\sigma}W^\rho p^\sigma$$

в) $[\varepsilon^{\mu\nu\rho\sigma} M_{\mu\nu} M_{\rho\sigma}, M^{\alpha\beta}] = 0$

$$[\varepsilon^{\mu\nu\rho\sigma} M_{\mu\nu} M_{\rho\sigma}, M^{\alpha\beta}] = \varepsilon^{\mu\nu\rho\sigma}\left([M_{\mu\nu}, M^{\alpha\beta}] M_{\rho\sigma} + M_{\mu\nu}[M_{\rho\sigma}, M^{\alpha\beta}]\right)$$

$$= i\varepsilon^{\mu\nu\rho\sigma}\Big(\big(\delta_\mu^\beta M_\nu{}^\alpha + \delta_\nu^\alpha M_\mu{}^\beta - \delta_\mu^\alpha M_\nu{}^\beta - \delta_\nu^\beta M_\mu{}^\alpha\big) M_{\rho\sigma}$$
$$+ M_{\mu\nu}\big(\delta_\rho^\beta M_\sigma{}^\alpha + \delta_\sigma^\alpha M_\rho{}^\beta - \delta_\rho^\alpha M_\sigma{}^\beta - \delta_\sigma^\beta M_\rho{}^\alpha\big)\Big)$$

$$= i\Big(\big(\varepsilon^{\beta\nu\rho\sigma} M_{\nu\mu} g^{\mu\alpha} + \varepsilon^{\mu\alpha\rho\sigma} M_{\mu\nu} g^{\nu\beta} - \varepsilon^{\alpha\nu\rho\sigma} M_{\nu\mu} g^{\mu\beta} - \varepsilon^{\mu\beta\rho\sigma} M_{\mu\nu} g^{\nu\alpha}\big) M_{\rho\sigma}$$
$$+ M_{\mu\nu}\big(\varepsilon^{\mu\nu\beta\sigma} M_{\sigma\rho} g^{\rho\alpha} + \varepsilon^{\mu\nu\rho\alpha} M_{\rho\sigma} g^{\sigma\beta} - \varepsilon^{\mu\nu\alpha\sigma} M_{\sigma\rho} g^{\rho\beta} - \varepsilon^{\mu\nu\rho\beta} M_{\rho\sigma} g^{\sigma\alpha}\big)\Big)$$

$$= iM_{\mu\nu}M_{\rho\sigma}\big(-\underbrace{\varepsilon^{\beta\nu\rho\sigma} g^{\mu\alpha}}_{1} + \underbrace{\varepsilon^{\mu\alpha\rho\sigma} g^{\nu\beta}}_{1} + \underbrace{\varepsilon^{\alpha\nu\rho\sigma} g^{\mu\beta}}_{2} - \underbrace{\varepsilon^{\mu\beta\rho\sigma} g^{\nu\alpha}}_{2}$$
$$- \underbrace{\varepsilon^{\mu\nu\beta\sigma} g^{\rho\alpha}}_{3} + \underbrace{\varepsilon^{\mu\nu\rho\alpha} g^{\sigma\beta}}_{3} + \varepsilon^{\mu\nu\alpha\sigma} g^{\rho\beta} - \underbrace{\varepsilon^{\mu\nu\rho\beta} g^{\sigma\alpha}}_{4}\big)$$

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$$\underbrace{\varepsilon^{\beta\nu\rho\sigma} g^{\mu\alpha}}_{1} + \underbrace{\varepsilon^{\mu\beta\nu\rho} g^{\sigma\alpha}}_{4} + \underbrace{\varepsilon^{\sigma\mu\beta\nu} g^{\rho\alpha}}_{3} + \underbrace{\varepsilon^{\rho\sigma\mu\beta} g^{\nu\alpha}}_{2} + \varepsilon^{\nu\rho\sigma\mu} g^{\beta\alpha} = 0$$
$$\underbrace{\varepsilon^{\mu\alpha\rho\sigma} g^{\nu\beta}}_{1} + \underbrace{\varepsilon^{\nu\mu\alpha\rho} g^{\sigma\beta}}_{3} + \varepsilon^{\sigma\nu\mu\alpha} g^{\rho\beta} + \varepsilon^{\rho\sigma\nu\mu} g^{\alpha\beta} + \underbrace{\varepsilon^{\alpha\rho\sigma\nu} g^{\mu\beta}}_{2} = 0$$

$$[\varepsilon^{\mu\nu\rho\sigma} M_{\mu\nu}M_{\rho\sigma}, M^{\alpha\beta}] = iM_{\mu\nu}M_{\rho\sigma}\left(\varepsilon^{\nu\rho\sigma\mu} g^{\beta\alpha} - \varepsilon^{\rho\sigma\nu\mu} g^{\alpha\beta}\right) = 0$$

Показати да $W^2|m,s,\vec{p}=0,\sigma\rangle = -m^2 s(s+1)|m,s,\vec{p}=0,\sigma\rangle$, где је вектор $|m,s,\vec{p}=0,\sigma\rangle$ вектор стања честице масе m, спина s, импулса $\vec{p}$ и пројекције спина на z-осу σ. Маса и спин класификују иредуцибилне репрезентације Поенкареове групе.

$$W^2 = -\frac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 + M_{\mu\sigma}M^{\nu\sigma}P^\mu P_\nu$$

$$\hat{P}^\mu|m,s,\vec{p}=0,\sigma\rangle = p^\mu|m,s,\vec{p}=0,\sigma\rangle = m\delta^\mu_0|m,s,\vec{p}=0,\sigma\rangle$$

$$P^2|m,s,\vec{p}=0,\sigma\rangle = P^\mu P_\mu|m,s,\vec{p}=0,\sigma\rangle = P^\mu m\delta^0_\mu|m,s,\vec{p}=0,\sigma\rangle =$$
$$= m^2\delta^\mu_0\delta^0_\mu|m,s,\vec{p}=0,\sigma\rangle = m^2|m,s,\vec{p}=0,\sigma\rangle$$

$$W^2|m,s,\vec{p}=0,\sigma\rangle = \left(-\frac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 + M_{\mu\sigma}M^{\nu\sigma}P^\mu P_\nu\right)|m,s,\vec{p}=0,\sigma\rangle$$
$$= -\frac{1}{2}m^2 M_{\mu\nu}M^{\mu\nu}|m,s,\vec{p}=0,\sigma\rangle + M_{\mu\sigma}M^{\nu\sigma}m^2\delta^\mu_0\delta^0_\nu|m,s,\vec{p}=0,\sigma\rangle$$
$$= \left[-\frac{1}{2}m^2\left(M_{0i}M^{0i} + M_{i0}M^{i0} + M_{ij}M^{ij}\right) + m^2 M_{0i}M^{0i}\right]|m,s,\vec{p}=0,\sigma\rangle$$
$$= -\frac{1}{2}m^2 M_{ij}M^{ij}|m,s,\vec{p}=0,\sigma\rangle$$

M_{ij} - генератори ротација; уведемо $J_i = \frac{1}{2}\varepsilon_{ijk}M_{jk}$ - оператор спина

$$[J_i,J_j] = \frac{1}{4}\varepsilon_{ikl}\varepsilon_{jnm}[M_{kl},M_{nm}] = \frac{i}{4}\varepsilon_{ikl}\varepsilon_{jnm}(\underbrace{g_{km}}_{-\delta_{km}}M_{ln} + \underbrace{g_{ln}}_{-\delta_{ln}}M_{km} - \underbrace{g_{kn}}_{-\delta_{kn}}M_{lm} - \underbrace{g_{lm}}_{-\delta_{lm}}M_{kn})$$
$$= -\frac{i}{4}\varepsilon_{ikl}\varepsilon_{jnk}M_{ln} - \frac{i}{4}\varepsilon_{ikl}\varepsilon_{jlm}M_{km} + \frac{i}{4}\varepsilon_{ikl}\varepsilon_{jkm}M_{lm} + \frac{i}{4}\varepsilon_{ikl}\varepsilon_{jnl}M_{kn}$$
$$= +\frac{i}{4}(\cancel{\delta_{ij}\delta_{ln}} - \delta_{in}\delta_{lj})M_{ln} + \frac{i}{4}(\cancel{\delta_{ij}\delta_{km}} - \delta_{im}\delta_{kj})M_{km} + \frac{i}{4}(\cancel{\delta_{ij}\delta_{lm}} - \delta_{im}\delta_{lj})M_{lm} + \frac{i}{4}(\cancel{\delta_{ij}\delta_{kn}} - \delta_{in}\delta_{kj})M_{kn}$$
$$= +\frac{i}{4}M_{ij} + \frac{i}{4}M_{ij} + \frac{i}{4}M_{ij} + \frac{i}{4}M_{ij} = iM_{ij}$$
$$i\varepsilon_{ijk}J_k = i\varepsilon_{ijk}\frac{1}{2}\varepsilon_{klm}M_{lm} = \frac{i}{2}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})M_{lm} = \frac{i}{2}M_{ij} + \frac{i}{2}M_{ij} = iM_{ij}$$

$\Rightarrow [J_i,J_j] = i\varepsilon_{ijk}J_k$ su(2) алгебра

Казимиров оператор за su(2) је $\vec{J}^2$ – својствене вредности су $s(s+1)$

$$\vec{J}^2 = J_iJ_i = \frac{1}{2}\varepsilon_{ijk}M_{jk}\frac{1}{2}\varepsilon_{ilm}M_{lm} = \frac{1}{4}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})M_{jk}M_{lm} = \frac{1}{4}M_{jk}M_{jk} - \frac{1}{4}M_{jk}M_{kj}$$
$$\vec{J}^2 = \frac{1}{2}M_{ij}M_{ij} = \frac{1}{2}M_{ij}M^{ij}$$

$$W^2|m,s,\vec{p}=0,\sigma\rangle = -\frac{1}{2}m^2M_{ij}M^{ij}|m,s,\vec{p}=0,\sigma\rangle = -m^2\vec{J}^2|m,s,\vec{p}=0,\sigma\rangle = \boxed{-m^2s(s+1)|m,s,\vec{p}=0,\sigma\rangle}$$

Показати да важи:

a) $[M_{\mu\nu}, W_\sigma] = i(g_{\nu\sigma}W_\mu - g_{\mu\sigma}W_\nu)$

$ЛС = [M_{\mu\nu}, W_\sigma] = [M_{\mu\nu}, \frac{1}{2}\varepsilon_{\sigma\alpha\beta\gamma}M^{\alpha\beta}P^\gamma] = \frac{1}{2}\varepsilon_{\sigma\alpha\beta\gamma}[M_{\mu\nu}, M^{\alpha\beta}]P^\gamma + \frac{1}{2}\varepsilon_{\sigma\alpha\beta\gamma}M^{\alpha\beta}[M_{\mu\nu}, P^\gamma]$

$= \frac{i}{2}\varepsilon_{\sigma\alpha\beta\gamma}(\delta_\mu^{\ \beta}M_\nu^{\ \alpha} + \delta_\nu^{\ \alpha}M_\mu^{\ \beta} - \delta_\mu^{\ \alpha}M_\nu^{\ \beta} - \delta_\nu^{\ \beta}M_\mu^{\ \alpha})P^\gamma + \frac{i}{2}\varepsilon_{\sigma\alpha\beta\gamma}M^{\alpha\beta}(\delta_\nu^{\ \gamma}P_\mu - \delta_\mu^{\ \gamma}P_\nu)$

$= \frac{i}{2}\varepsilon_{\sigma\alpha\mu\gamma}M_\nu^{\ \alpha}P^\gamma + \frac{i}{2}\varepsilon_{\sigma\nu\beta\gamma}M_\mu^{\ \beta}P^\gamma - \frac{i}{2}\varepsilon_{\sigma\mu\beta\gamma}M_\nu^{\ \beta}P^\gamma - \frac{i}{2}\varepsilon_{\sigma\alpha\nu\gamma}M_\mu^{\ \alpha}P^\gamma + \frac{i}{2}\varepsilon_{\sigma\alpha\beta\nu}M^{\alpha\beta}P_\mu - \frac{i}{2}\varepsilon_{\sigma\alpha\beta\mu}M^{\alpha\beta}P_\nu$

$= -i\varepsilon_{\sigma\mu\alpha\beta}M_\nu^{\ \alpha}P^\beta + i\varepsilon_{\sigma\nu\alpha\beta}M_\mu^{\ \alpha}P^\beta + \frac{i}{2}\varepsilon_{\sigma\nu\alpha\beta}M^{\alpha\beta}P_\mu - \frac{i}{2}\varepsilon_{\sigma\mu\alpha\beta}M^{\alpha\beta}P_\nu$

$ДС = i(g_{\nu\sigma}W_\mu - g_{\mu\sigma}W_\nu) = \frac{i}{2}\varepsilon_{\mu\alpha\beta\gamma}g_{\nu\sigma}M^{\alpha\beta}P^\gamma - \frac{i}{2}\varepsilon_{\nu\alpha\beta\gamma}g_{\mu\sigma}M^{\alpha\beta}P^\gamma$

1° $\mu=\nu$ $ЛС=0$ $ДС=0$ – задовољено

2° $\mu\neq\nu$ $\sigma=\mu$ $ЛС = i\varepsilon_{\mu\nu\alpha\beta}M_\mu^{\ \alpha}P^\beta + \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}M^{\alpha\beta}P_\mu = i\varepsilon_{\mu\nu\varphi\chi}M_\mu^{\ \varphi}P^\chi + i\varepsilon_{\mu\nu\chi\varphi}M_\mu^{\ \chi}P^\varphi$

$\{\mu,\nu,\varphi,\chi\}=\{0,1,2,3\}$

$+ \frac{i}{2}\varepsilon_{\mu\nu\varphi\chi}M^{\varphi\chi}P_\mu \cdot 2$

$ЛС = i\varepsilon_{\mu\nu\varphi\chi}(M_\mu^{\ \varphi}P^\chi - M_\mu^{\ \chi}P^\varphi + M^{\varphi\chi}P_\mu)$

$ДС = -\frac{i}{2}\varepsilon_{\nu\alpha\beta\gamma}g_{\mu\mu}M^{\alpha\beta}P^\gamma = -\frac{i}{2}\varepsilon_{\nu\mu\varphi\chi}g_{\mu\mu}M^{\mu\varphi}P^\chi - \frac{i}{2}\varepsilon_{\nu\mu\chi\varphi}g_{\mu\mu}M^{\mu\chi}P^\varphi - \frac{i}{2}\varepsilon_{\nu\varphi\mu\chi}g_{\mu\mu}M^{\varphi\mu}P^\chi - \frac{i}{2}\varepsilon_{\nu\varphi\chi\mu}M^{\varphi\chi}P^\mu g_{\mu\mu}$

$\{\alpha,\beta,\gamma\}=\{\mu,\varphi,\chi\}$

$- \frac{i}{2}\varepsilon_{\nu\chi\mu\varphi}g_{\mu\mu}M^{\chi\mu}P^\varphi - \frac{i}{2}\varepsilon_{\nu\chi\varphi\mu}g_{\mu\mu}M^{\chi\varphi}P^\mu$

$= +\frac{i}{2}\varepsilon_{\mu\nu\varphi\chi}M_\mu^{\ \varphi}P^\chi - \frac{i}{2}\varepsilon_{\mu\nu\varphi\chi}M_\mu^{\ \chi}P^\varphi - \frac{i}{2}\varepsilon_{\mu\nu\varphi\chi}M^\varphi_{\ \mu}P^\chi + \frac{i}{2}\varepsilon_{\mu\nu\varphi\chi}M^{\varphi\chi}P_\mu + \frac{i}{2}\varepsilon_{\mu\nu\varphi\chi}M^\chi_{\ \mu}P^\varphi - \frac{i}{2}\varepsilon_{\mu\nu\varphi\chi}M^{\chi\varphi}P_\mu$

$= i\varepsilon_{\mu\nu\varphi\chi}(M_\mu^{\ \varphi}P^\chi - M_\mu^{\ \chi}P^\varphi + M^{\varphi\chi}P_\mu)$ $ЛС = ДС$

3° $\mu\neq\nu$ $\sigma=\nu$ исто као 2°

4° $\mu\neq\nu\neq\sigma$ $\{\mu,\nu,\sigma,\varphi\}=\{0,1,2,3\}$

$ЛС = -i\varepsilon_{\sigma\mu\nu\varphi}\underbrace{M_\nu^{\ \nu}}_{0}P^\varphi - i\varepsilon_{\sigma\mu\varphi\nu}M_\nu^{\ \varphi}P^\nu + i\varepsilon_{\sigma\nu\mu\varphi}\underbrace{M_\mu^{\ \mu}}_{0}P^\varphi + i\varepsilon_{\sigma\nu\varphi\mu}M_\mu^{\ \varphi}P^\mu + \frac{i}{2}\varepsilon_{\sigma\nu\mu\varphi}M^{\mu\varphi}P_\mu \cdot 2$

$- \frac{i}{2}\varepsilon_{\sigma\mu\nu\varphi}M^{\nu\varphi}P_\nu \cdot 2$

$= i\varepsilon_{\mu\nu\sigma\varphi}(\cancel{M^{\nu\varphi}P_\nu} + \cancel{M^{\mu\varphi}P_\mu} - \cancel{M^{\mu\varphi}P_\mu} - \cancel{M^{\nu\varphi}P_\nu}) = 0$

$ДС = 0$ задовољена једнакост

б) $[W_\mu, W_\nu] = -i\varepsilon_{\mu\nu\sigma\rho} W^\sigma P^\rho$

$$\text{ЛС} = [W_\mu, W_\nu] = \frac14 \varepsilon_{\mu\rho\sigma\lambda}\varepsilon_{\nu\alpha\beta\gamma}[M^{\rho\sigma}P^\lambda, M^{\alpha\beta}P^\gamma] = \frac14 \varepsilon_{\mu\rho\sigma\lambda}\varepsilon_{\nu\alpha\beta\gamma}\left([M^{\rho\sigma}P^\lambda, M^{\alpha\beta}]P^\gamma + M^{\alpha\beta}[M^{\rho\sigma}P^\lambda, P^\gamma]\right)$$

$$= \frac14 \varepsilon_{\mu\rho\sigma\lambda}\varepsilon_{\nu\alpha\beta\gamma}\left([M^{\rho\sigma}, M^{\alpha\beta}]P^\lambda P^\gamma + M^{\rho\sigma}[P^\lambda, M^{\alpha\beta}]P^\gamma + M^{\alpha\beta}[M^{\rho\sigma}, P^\gamma]P^\lambda + M^{\alpha\beta}M^{\rho\sigma}\underbrace{[P^\lambda, P^\gamma]}_{0}\right)$$

$$= \frac{i}{4}\varepsilon_{\mu\rho\sigma\lambda}\varepsilon_{\nu\alpha\beta\gamma}\left((g^{\rho\beta}M^{\sigma\alpha} + g^{\sigma\alpha}M^{\rho\beta} - g^{\rho\alpha}M^{\sigma\beta} - g^{\sigma\beta}M^{\rho\alpha})P^\lambda P^\gamma + M^{\rho\sigma}(\underbrace{g^{\lambda\alpha}P^\beta}_{\text{антисим }\beta\gamma} - \underbrace{g^{\lambda\beta}P^\alpha}_{\text{антисим }\alpha\gamma})P^\gamma + M^{\alpha\beta}(\underbrace{g^{\sigma\gamma}P^\rho}_{\text{антисим }\rho\lambda} - \underbrace{g^{\rho\gamma}P^\sigma}_{\text{антисим }\sigma\lambda})P^\lambda\right)$$

$$= \frac{i}{4}\varepsilon_{\mu\rho\sigma\lambda}\varepsilon_{\nu\alpha}{}^{\rho}{}_{\gamma}M^{\sigma\alpha}P^\lambda P^\gamma + \underbrace{\frac{i}{4}\varepsilon_{\mu\rho\sigma\lambda}\varepsilon_{\nu}{}^{\sigma}{}_{\beta\gamma}M^{\rho\beta}P^\lambda P^\gamma}_{\beta\to\alpha,\ \sigma\leftrightarrow\rho} - \underbrace{\frac{i}{4}\varepsilon_{\mu\rho\sigma\lambda}\varepsilon_{\nu}{}^{\rho}{}_{\beta\gamma}M^{\sigma\beta}P^\lambda P^\gamma}_{\beta\to\alpha} - \underbrace{\frac{i}{4}\varepsilon_{\mu\rho\sigma\lambda}\varepsilon_{\nu\alpha}{}^{\sigma}{}_{\gamma}M^{\rho\alpha}P^\lambda P^\gamma}_{\rho\leftrightarrow\sigma}$$

$$= i\underbrace{\varepsilon_{\mu\rho\sigma\lambda}}_{\mu\sigma\lambda}\underbrace{\varepsilon_{\nu\alpha}{}^{\rho}{}_{\gamma}}_{\nu\alpha\gamma}M^{\sigma\alpha}P^\lambda P^\gamma = +i\left(g_{\mu\nu}\underbrace{g_{\sigma\alpha}}_{0}g_{\lambda\gamma} + g_{\mu\gamma}g_{\sigma\nu}g_{\lambda\alpha} + g_{\mu\alpha}g_{\sigma\gamma}g_{\lambda\nu} - g_{\mu\alpha}g_{\sigma\nu}g_{\lambda\gamma} - g_{\mu\gamma}\underbrace{g_{\sigma\alpha}}_{0}g_{\lambda\nu} - g_{\mu\nu}g_{\sigma\gamma}g_{\lambda\alpha}\right)M^{\sigma\alpha}P^\lambda P^\gamma$$

$$= iM_{\nu\lambda}P^\lambda P_\mu + iM_{\gamma\mu}P_\nu P^\gamma - iM_{\nu\mu}P^2 - ig_{\mu\nu}\cancel{\underbrace{M_{\rho\sigma}}_{\text{антисим }\rho\sigma}\underbrace{P^\sigma P^\rho}_{\text{сим }\rho\sigma}}$$

$$= -iM_{\mu\rho}P_\nu P^\rho + iM_{\nu\rho}P_\mu P^\rho + iM_{\mu\nu}P^2$$

$$\text{ДС} = -i\varepsilon_{\mu\nu\sigma\rho}W^\sigma P^\rho = -i\frac12 \underbrace{\varepsilon_{\mu\nu\sigma\rho}}_{(\mu\nu\rho)}\underbrace{\varepsilon^{\sigma\alpha\beta\gamma}}_{(\alpha\beta\gamma)}M_{\alpha\beta}P_\gamma P^\rho = +\frac{i}{2}\left(\delta_\mu^\alpha\delta_\nu^\beta\delta_\rho^\gamma + \delta_\mu^\gamma\delta_\nu^\alpha\delta_\rho^\beta + \delta_\mu^\beta\delta_\nu^\gamma\delta_\rho^\alpha - \delta_\mu^\beta\delta_\nu^\alpha\delta_\rho^\gamma - \delta_\mu^\gamma\delta_\nu^\beta\delta_\rho^\alpha - \delta_\mu^\alpha\delta_\nu^\gamma\delta_\rho^\beta\right)M_{\alpha\beta}P_\gamma P^\rho$$

$$= \frac{i}{2}M_{\mu\nu}P^2 + \frac{i}{2}M_{\nu\rho}P_\mu P^\rho + \frac{i}{2}M_{\rho\mu}P_\nu P^\rho - \frac{i}{2}M_{\nu\mu}P^2 - \frac{i}{2}M_{\rho\nu}P_\mu P^\rho - \frac{i}{2}M_{\mu\rho}P_\nu P^\rho$$

$$= -iM_{\mu\rho}P_\nu P^\rho + iM_{\nu\rho}P_\mu P^\rho + iM_{\mu\nu}P^2 \qquad \boxed{\text{ЛС} = \text{ДС}}$$

в) $[\varepsilon^{\mu\nu\rho\delta}M_{\mu\nu}M_{\rho\delta}, M_{\alpha\beta}] = \varepsilon^{\mu\nu\rho\delta}[M_{\mu\nu}, M_{\alpha\beta}]M_{\rho\delta} + \varepsilon^{\mu\nu\rho\delta}M_{\mu\nu}[M_{\rho\delta}, M_{\alpha\beta}]$

$$= i\varepsilon^{\mu\nu\rho\delta}(g_{\mu\beta}M_{\nu\alpha} + g_{\nu\alpha}M_{\mu\beta} - g_{\mu\alpha}M_{\nu\beta} - g_{\nu\beta}M_{\mu\alpha})M_{\rho\delta} + i\varepsilon^{\mu\nu\rho\delta}M_{\mu\nu}(g_{\rho\beta}M_{\delta\alpha} + g_{\delta\alpha}M_{\rho\beta} - g_{\rho\alpha}M_{\delta\beta} - g_{\delta\beta}M_{\rho\alpha})$$

$$= i(\varepsilon_\beta{}^{\nu\rho\delta}M_{\nu\alpha} + \varepsilon^\mu{}_\alpha{}^{\rho\delta}M_{\mu\beta} - \varepsilon_\alpha{}^{\nu\rho\delta}M_{\nu\beta} - \varepsilon^\mu{}_\beta{}^{\rho\delta}M_{\mu\alpha})M_{\rho\delta} + iM_{\mu\nu}(\varepsilon^{\mu\nu}{}_\beta{}^\delta M_{\delta\alpha} + \varepsilon^{\mu\nu\rho}{}_\alpha M_{\rho\beta} - \varepsilon^{\mu\nu}{}_\alpha{}^\delta M_{\delta\beta} - \varepsilon^{\mu\nu\rho}{}_\beta M_{\rho\alpha})$$

$$= 2i\varepsilon_\beta{}^{\mu\rho\delta}M_{\mu\alpha}M_{\rho\delta} - 2i\varepsilon_\alpha{}^{\mu\rho\delta}M_{\mu\beta}M_{\rho\delta} + 2i\varepsilon_\beta{}^{\mu\rho\delta}M_{\rho\delta}M_{\mu\alpha} - 2i\varepsilon_\alpha{}^{\mu\rho\delta}M_{\rho\delta}M_{\mu\beta}$$

$$= 4i\varepsilon_\beta{}^{\mu\rho\delta}M_{\mu\alpha}M_{\rho\delta} - 2\varepsilon_\beta{}^{\mu\rho\delta}(g_{\rho\alpha}M_{\delta\mu} + \underbrace{g_{\delta\mu}}_{0}M_{\rho\alpha} - \underbrace{g_{\rho\mu}}_{0}M_{\delta\alpha} - g_{\delta\alpha}M_{\rho\mu})$$

$$- 4i\varepsilon_\alpha{}^{\mu\rho\delta}M_{\mu\beta}M_{\rho\delta} + 2\varepsilon_\alpha{}^{\mu\rho\delta}(g_{\rho\beta}M_{\delta\mu} + \underbrace{g_{\delta\mu}}_{0}M_{\rho\beta} - \underbrace{g_{\rho\mu}}_{0}M_{\delta\beta} - g_{\delta\beta}M_{\rho\mu})$$

$$= 4i\varepsilon_\beta{}^{\mu\rho\delta}M_{\mu\alpha}M_{\rho\delta} - 4i\varepsilon_\alpha{}^{\mu\rho\delta}M_{\mu\beta}M_{\rho\delta} + 8\varepsilon_{\alpha\beta}{}^{\mu\nu}M_{\mu\nu}$$

1° $\alpha = \beta$ комут. $= 0$

2° $\alpha \neq \beta$ $\{\alpha, \beta, \gamma, \delta\} = \{0,1,2,3\}$

$$[\varepsilon^{\mu\nu\rho\delta}M_{\mu\nu}M_{\rho\delta}, M_{\alpha\beta}] = 4i\underbrace{\varepsilon_\beta{}^{\mu\rho\delta}M_{\mu\alpha}M_{\rho\delta}}_{\mu\neq\alpha,\ \mu\neq\beta\ \ \mu=\{\gamma,\delta\}} - 4i\underbrace{\varepsilon_\alpha{}^{\mu\rho\delta}M_{\mu\beta}M_{\rho\delta}}_{\mu\neq\alpha,\ \mu\neq\beta\ \ \mu=\{\gamma,\delta\}} + 8\underbrace{\varepsilon_{\alpha\beta}{}^{\mu\nu}M_{\mu\nu}}_{\{\mu,\nu\}=\{\gamma,\delta\}}$$

$$= 4i\varepsilon_\beta{}^{\gamma\alpha\delta}M_{\gamma\alpha}M_{\alpha\delta}\cdot 2 + 4i\varepsilon_\beta{}^{\delta\alpha\gamma}M_{\delta\alpha}M_{\alpha\gamma}\cdot 2 - 4i\varepsilon_\alpha{}^{\gamma\beta\delta}M_{\gamma\beta}M_{\beta\delta}\cdot 2 - 4i\varepsilon_\alpha{}^{\delta\beta\gamma}M_{\delta\beta}M_{\beta\gamma}\cdot 2 + 16\varepsilon_{\alpha\beta\gamma\delta}M^{\gamma\delta}$$

$$= 8i\varepsilon_{\alpha\beta\gamma\delta}(-M^{\alpha\gamma}M_\alpha{}^\delta + M_\alpha{}^\delta M^{\alpha\gamma} + M^{\gamma\beta}M_\beta{}^\delta - M_\beta{}^\delta M^{\gamma\beta}) + 16\varepsilon_{\alpha\beta\gamma\delta}M^{\gamma\delta}$$

$$= 8i\varepsilon_{\alpha\beta\gamma\delta}([M_\alpha{}^\delta, M^{\alpha\gamma}] + [M^{\gamma\beta}, M_\beta{}^\delta]) + 16\varepsilon_{\alpha\beta\gamma\delta}M^{\gamma\delta} = -8\varepsilon_{\alpha\beta\gamma\delta}(\underbrace{\delta_\alpha^\gamma}_{0}M^{\delta\alpha} + \underbrace{g^{\delta\alpha}}_{0}M_\alpha{}^\gamma - \underbrace{\delta_\alpha^\alpha}_{1}M^{\delta\gamma} - \underbrace{g^{\delta\gamma}}_{0}M_\alpha{}^\alpha$$

$$+ \underbrace{g^{\gamma\delta}}_{0}M^\beta{}_\beta + \underbrace{\delta^\beta_\beta}_{+1}M^{\gamma\delta} - \underbrace{\delta^\gamma_\beta}_{0}M^{\beta\delta} - \underbrace{g^{\beta\delta}}_{0}M^\gamma{}_\beta) + 16\varepsilon_{\alpha\beta\gamma\delta}M^{\gamma\delta} = 8\varepsilon_{\alpha\beta\gamma\delta}M^{\delta\gamma} - 8\varepsilon_{\alpha\beta\gamma\delta}M^{\gamma\delta} + 16\varepsilon_{\alpha\beta\gamma\delta}M^{\gamma\delta} = 0$$

$$\Rightarrow \quad [\varepsilon^{\mu\nu\rho\delta}M_{\mu\nu}M_{\rho\delta}, M_{\alpha\beta}] = 0$$

Стандардни импулс за честицу масе m је $(m,0,0,0)$, док је за безмасену честицу $(k,0,0,k)$. Показати да је мала група за масене честице $SU(2)$, а за безмасене $E(2)$ (група транслација и ротација у дводимензионој равни).

Стандардни импулс $\bar{p}$

Вигнерове ротације – трансф. из $\mathcal{P}$ које чувају инваријантним $\bar{p}$

$$R\bar{p} = \bar{p}$$

Вигнерове ротације чине групу

1° R_1 и R_2 – Вигнерове ротације $\Rightarrow$ $R_1\bar{p} = \bar{p}$ и $R_2\bar{p} = \bar{p}$

$(R_1R_2)\bar{p} = R_1(R_2\bar{p}) = R_1\bar{p} = \bar{p}$ $\Rightarrow$ R_1R_2 – Вигнерова ротација

$\Rightarrow$ затвореност

2° јединични елемент I $\quad R\cdot I\bar{p} = R\bar{p} = \bar{p} = I(R\bar{p}) = I\cdot\bar{p} = \bar{p}$

$\Rightarrow \boxed{RI = IR = R}$

3° инверз $\quad R^{-1}R = RR^{-1} = I \qquad R\bar{p} = \bar{p} \Rightarrow \bar{p} = R^{-1}\bar{p}$ па је R^{-1} Вигнерова ротација

4° асоцијативност нагле важи у $\mathcal{P}$.

1) масене честице $\bar{p} = (m,0,0,0)$

$$\bar{p} = \begin{pmatrix} m\\0\\0\\0\end{pmatrix} = R\begin{pmatrix} m\\0\\0\\0\end{pmatrix} = \begin{pmatrix} 1 & \omega^0{}_1 & \omega^0{}_2 & \omega^0{}_3\\ \omega^1{}_0 & 1 & \omega^1{}_2 & \omega^1{}_3\\ \omega^2{}_0 & \omega^2{}_1 & 1 & \omega^2{}_3\\ \omega^3{}_0 & \omega^3{}_1 & \omega^3{}_2 & 1\end{pmatrix}\begin{pmatrix} m\\0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix} 0 & \omega^0{}_1 & \omega^0{}_2 & \omega^0{}_3\\ \omega^1{}_0 & 0 & \omega^1{}_2 & \omega^1{}_3\\ \omega^2{}_0 & \omega^2{}_1 & 0 & \omega^2{}_3\\ \omega^3{}_0 & \omega^3{}_1 & \omega^3{}_2 & 0\end{pmatrix}\begin{pmatrix} m\\0\\0\\0\end{pmatrix} = 0 \Rightarrow \begin{matrix}\omega^1{}_0 m = 0\\ \omega^2{}_0 m = 0\\ \omega^3{}_0 m = 0\end{matrix}$$

$\Rightarrow \omega_{01} = \omega_{02} = \omega_{03} = 0 \qquad \omega_{ij} \neq 0 \qquad M^{12}, M^{23}, M^{31}$ генератори ротација

мала група је $SO(3)$ $\qquad \overline{SO(3)} = SU(2)$

$J_i = \frac{1}{2}\varepsilon_{ijk}M_{jk} \qquad [J_i, J_j] = i\varepsilon_{ijk}J_k \quad SU(2)$ алгебра

2) безмасене честице $\bar{p} = (k,0,0,k)$

$$\bar{p} = \begin{pmatrix} k\\0\\0\\k\end{pmatrix} = R\begin{pmatrix} k\\0\\0\\k\end{pmatrix} \Rightarrow (R-I)\begin{pmatrix} k\\0\\0\\k\end{pmatrix} = 0 \qquad \begin{pmatrix} 0 & \omega^0{}_1 & \omega^0{}_2 & \omega^0{}_3\\ \omega^1{}_0 & 0 & \omega^1{}_2 & \omega^1{}_3\\ \omega^2{}_0 & \omega^2{}_1 & 0 & \omega^2{}_3\\ \omega^3{}_0 & \omega^3{}_1 & \omega^3{}_2 & 0\end{pmatrix}\begin{pmatrix} k\\0\\0\\k\end{pmatrix} = 0 \Rightarrow \begin{matrix}\omega^0{}_3 k = 0\\ (\omega^1{}_0 + \omega^1{}_3)k = 0\\ (\omega^2{}_0 + \omega^2{}_3)k = 0\\ \omega^3{}_0 k = 0\end{matrix}$$

$\omega_{03} = \omega_{01} - \omega_{13} = \omega_{02} - \omega_{23} = 0 \qquad \omega_{12}$ је произвољно

$R = e^{-\frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}}; \quad -\frac{i}{2}M_{\mu\nu}\omega^{\mu\nu} = -\frac{i}{2}M_{01}\omega^{01}\cdot 2 - iM_{02}\omega^{02} - iM_{03}\omega^{03} - iM_{12}\omega^{12} - iM_{13}\omega^{13} - iM_{23}\omega^{23}$

$= -iM^{01}\omega_{01} - iM^{13}\omega_{01} - iM^{02}\omega_{02} - iM^{23}\omega_{02} - iM^{12}\omega_{12}$

$= -i(M^{01} + M^{13})\omega_{01} - i(M^{02} + M^{23})\omega_{02} - iM^{12}\omega_{12}$

генератори су: $M^{01} + M^{13}$, $M^{02} + M^{23}$ и M^{12}

$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}M^{\nu\rho}p^\sigma = \frac{1}{2}\varepsilon_{\mu\nu\rho 0}M^{\nu\rho}k + \frac{1}{2}\varepsilon_{\mu\nu\rho 3}M^{\nu\rho}k$ (кад делује на $\bar{p}$)

$\boxed{W_0} = \frac{1}{2}\varepsilon_{0\nu\rho 3}M^{\nu\rho}\cdot k = \varepsilon_{0123}M^{12}k = \boxed{-kM^{12}}$; $\boxed{W_1} = \varepsilon_{1230}M^{23}k + \varepsilon_{1023}M^{02}k = \boxed{(M^{23} + M^{02})k}$

$\boxed{W_2} = \varepsilon_{2130}M^{13}k + \varepsilon_{2013}M^{01}k = \boxed{-kM^{13} - kM^{01}}$; $W_3 = \varepsilon_{3120}M^{12}k = M^{12}k$; генер. $\frac{W_0}{k}, W_1, W_2$

$[W_1, W_2] = -k^2[M^{23} + M^{02}, M^{13} + M^{01}] = -ik^2(+M^{21} - M^{21}) = 0$; $[\frac{W_0}{k}, W_1] = -k[M^{12}, M^{23} + M^{02}] = -ik(+M^{13} + M^{10}) = -iW_2$

$[\frac{W_0}{k}, W_2] = k[M^{12}, M^{13} + M^{01}] = ik(M^{23} - M^{20}) = iW_1$ $\qquad$ Ово су ком. рел. $E(2)$ групе!