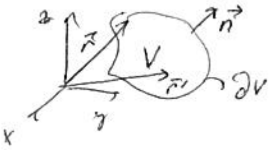
Метод Гринових фја

деф. $\Delta' G(\vec r,\vec r') = -\frac{1}{\varepsilon_0}\delta(\vec r-\vec r')$

Партикуларно решење $G_p(\vec r,\vec r') = \frac{1}{4\pi\varepsilon_0}\frac{1}{|\vec r-\vec r'|}$ ← потенцијал јединичног наелектрисања које се налази у $\vec r'$

Хомогено решење $G_h(\vec r,\vec r')$ задовољава $\Delta' G_h(\vec r,\vec r')=0$

2. Гринов идентитет: $\int_V d^3r\left(\phi \underbrace{\nabla^2}_{\Delta}\psi - \psi\nabla^2\phi\right) = \oint_{\partial V}(\phi\nabla\psi-\psi\nabla\phi)\,d\vec S$



за $\phi(\vec r)=V(\vec r)$; $\psi(\vec r) = G(\vec r,\vec r')$

$$\int dV'\Big(V(\vec r')\underbrace{\nabla'^2 G(\vec r,\vec r')}_{-\frac{1}{\varepsilon_0}\delta(\vec r-\vec r')} - G(\vec r,\vec r')\underbrace{\Delta' V(\vec r')}_{-\frac{1}{\varepsilon_0}\rho(\vec r')}\Big) = \oint_{\partial V'}\Big(V(\vec r')\nabla' G(\vec r,\vec r') - G(\vec r,\vec r')\nabla' V(\vec r')\Big)d\vec S'$$

$$-\frac{1}{\varepsilon_0}V(\vec r) + \frac{1}{\varepsilon_0}\int dV' G(\vec r,\vec r')\rho(\vec r') = \oint_{\partial V'}\Big(V(\vec r')\frac{\partial G}{\partial n'} - G(\vec r,\vec r')\frac{\partial V}{\partial n'}\Big)dS'$$

$$V(\vec r) = \underbrace{\int dV'\rho(\vec r')G(\vec r,\vec r')}_{\text{наелектрисања унутар } V} + \underbrace{\varepsilon_0\oint dS'\Big(G(\vec r,\vec r')\frac{\partial V(\vec r')}{\partial n'} - V(\vec r')\frac{\partial G(\vec r,\vec r')}{\partial n'}\Big)}_{\text{наелектрисања изван } V}$$

$G_{моје} = \frac{1}{4\pi\varepsilon_0}G_{Воја}$

G није једнозначно одређено, узимамо граничне услове који дају згодне Г.Ф.

1° Дирихлеов гранични услов $G_D(\vec r,\vec r')\big|_{\vec r\in\partial V}=0$

2° Нојманов гранични услов $\frac{\partial G_N}{\partial n}\Big|_S = -\frac{1}{\varepsilon_0 S}$

не може да се узме $\frac{\partial G_N}{\partial n}=0$ јер: $\Delta' G_N(\vec r,\vec r') = -\frac{1}{\varepsilon_0}\delta(\vec r-\vec r') \quad / \int d^3r'$

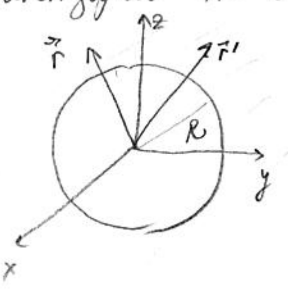
$\int d^3r' \,\mathrm{div}'\,\mathrm{grad}'\, G_N(\vec r,\vec r') = -\frac{1}{\varepsilon_0}\int d^3r'\delta(\vec r-\vec r') = -\frac{1}{\varepsilon_0}$ $\vec r\in$ области интеграције

$= \int d\vec S'\,\mathrm{grad}\, G_N(\vec r,\vec r') = \int dS'\frac{\partial G_N}{\partial n'} \Rightarrow \frac{\partial G_N}{\partial n'} = -\frac{1}{\varepsilon_0 S}$ $\boxed{\vec n\nabla' \equiv \frac{\partial}{\partial n'}}$

1° $V(\vec r) = \int d^3r'\rho(\vec r')G_D(\vec r,\vec r') - \varepsilon_0\oint dS'\, V(\vec r')\frac{\partial G_D(\vec r,\vec r')}{\partial n'}$

2° $V(\vec r) = \int d^3r'\rho(\vec r')G_N(\vec r,\vec r') + \varepsilon_0\oint dS'\, G_N(\vec r,\vec r')\frac{\partial V(\vec r')}{\partial n'} + \underbrace{\frac{1}{S}\int_{\partial V} dS'\, V(\vec r')}_{\langle V(\vec r')\rangle|_S}$

Наћи Гринову фју у области изван сфере полупречника R која задовољава Дирихлеов гранични услов на површини сфере. Користећи се овим резултатом одредити потенцијал на z-оси ако је потенцијал сфере $\phi(r=R) = \begin{cases}V_0, & 0\le\theta\le\frac{\pi}{2}\\ -V_0, & \frac{\pi}{2}\le\theta\le\pi\end{cases}$



$G(\vec r,\vec r') = \frac{1}{4\pi\varepsilon_0}\frac{1}{|\vec r-\vec r'|} + F(\vec r,\vec r')$; $\Delta F(\vec r,\vec r')=0$, F – тражимо да задовољи гран. услов.

$G_D(\vec r=R\vec e_r,\vec r') = G_D(\vec r,\vec r'=R\vec e_{r'}) = 0$

↑ симетрично по $\vec r,\vec r'$

Применимо метод ликова – да анулирамо Г.ф. на површини сфере, лик је унутар сфере, на растојању $\frac{R^2}{r'}$ од центра, и наелектрисање му је $-\frac{R}{r'}$

$$G_D(\vec{r},\vec{r}') = \frac{1}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi\varepsilon_0}\frac{-\frac{R}{r'}}{|\vec{r}-\frac{R^2}{r'}\vec{e}_{r'}|}$$

$\Delta F = 0$ јер је то потенцијал од наелектрисања изван области!

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{R}{r'}\frac{1}{\sqrt{r^2+\frac{R^4}{r'^2}-2r\frac{R^2}{r'}\cos\gamma}}\right)$$

$\gamma = \measuredangle(\vec{r},\vec{r}')$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{R}{\sqrt{r'^2r^2+R^4-2rr'R^2\cos\gamma}}\right)$$

$$V(\vec{r}=z\vec{e}_z) = \underbrace{\int dV'\,\underset{0}{\underset{\shortparallel}{\rho(\vec{r}')}}\,G(\overset{z\vec{e}_z}{\vec{r}},\vec{r}')}_{0} - \varepsilon_0\oint dS'\,V(\vec{r}')\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}$$

$$\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\bigg|_{\partial V'} \underset{\boxed{\vec{n}'=-\vec{e}_{r'}}}{=} -\frac{\partial G(\vec{r},\vec{r}')}{\partial r'}\bigg|_{\partial V'} = -\frac{1}{4\pi\varepsilon_0}\left(-\frac{1}{2}\right)\left[\frac{2R-2r\cos\gamma}{(r^2+R^2-2rR\cos\gamma)^{3/2}} - \frac{R(2Rr^2-2rR^2\cos\gamma)}{(R^2r^2+R^4-2rR^3\cos\gamma)^{3/2}}\right]$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{-R+\cancel{r\cos\gamma}+\frac{r^2}{R}-\cancel{r\cos\gamma}}{(r^2+R^2-2rR\cos\gamma)^{3/2}} = -\frac{1}{4\pi\varepsilon_0}\frac{r^2-R^2}{R(r^2+R^2-2rR\cos\gamma)^{3/2}}$$

$$V(\vec{r}=z\vec{e}_z) = +\cancel{\varepsilon_0}\frac{1}{4\pi\cancel{\varepsilon_0}}\frac{z^2-R^2}{R}\left[\int_0^{\pi/2}\int_0^{2\pi}R^2\sin\theta'\,d\theta'\,d\varphi'\,V_0\frac{1}{(z^2+R^2-2zR\cos\theta)^{3/2}} - \int_{\pi/2}^{\pi}\int_0^{2\pi}R^2\sin\theta'\,d\theta'\,d\varphi'\,V_0\frac{1}{(z^2+R^2-2zR\cos\theta)^{3/2}}\right]$$

$$= +\frac{1}{4\cancel{\pi}}\frac{z^2-R^2}{R}R^2V_0\,2\cancel{\pi}\left(\int_0^{\pi/2}\frac{d(z^2+R^2-2zR\cos\theta)}{(z^2+R^2-2zR\cos\theta)^{3/2}}\frac{1}{2z\cancel{R}} - \int_{\pi/2}^{\pi}\frac{d(z^2+R^2-2zR\cos\theta)}{(z^2+R^2-2zR\cos\theta)^{3/2}}\frac{1}{2z\cancel{R}}\right)$$

$$= +\frac{(z^2-R^2)V_0}{4z}(-2)\left(\frac{1}{\sqrt{z^2+R^2-2zR\cos\theta}}\bigg|_0^{\pi/2} - \frac{1}{\sqrt{z^2+R^2-2zR\cos\theta}}\bigg|_{\pi/2}^{\pi}\right)$$

$$= -\frac{(z^2-R^2)V_0}{2z}\left(\frac{1}{\sqrt{z^2+R^2}} - \frac{1}{z-R} - \frac{1}{z+R} + \frac{1}{\sqrt{z^2+R^2}}\right) = -\frac{(z^2-R^2)V_0}{2z}\left(\frac{2}{\sqrt{z^2+R^2}} - \frac{2z}{z^2-R^2}\right)$$

$$= \boxed{V_0\left(1-\frac{z^2-R^2}{z\sqrt{z^2+R^2}}\right)}$$

У делу простора $z>0$ са Дирихлеовим граничним условом на $z=0$ равни и у бесконачности наћи:

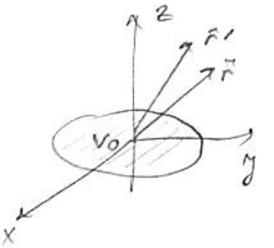
а) одговарајућу Гринову фју

б) Ако је потенцијал у $z=0$ равни дат са $\phi = \begin{cases} V_0, & r_c<a \\ 0, & r_c>a \end{cases}$ одредити потенцијал у делу $r_c, \varphi, z>0$

в) Показати да је дуж осе симетрије круга потенцијал $\phi = V_0\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$

г) Показати да је на великим растојањима $r_c^2+z^2 \gg a^2$ потенцијал:

$$\phi = \frac{V_0 a^2}{2}\frac{z}{(r_c^2+z^2)^{3/2}}\left(1 - \frac{3a^2}{4(r_c^2+z^2)} + \frac{5}{8}\frac{3r^2a^2+a^4}{(r_c^2+z^2)^2} + \dots\right)$$



$$G(\vec r, \vec r') = \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{1}{|\vec r - \vec r'|}}_{\text{потенцијал наел. у } \vec r'} + \underbrace{F(\vec r, \vec r')}_{\text{треба да анулира Г.ф. на равни и у } \infty.}$$

Метод ликова: лик наелектрисања 1 у $\vec r' = r_c'\vec e_{r_c'} + z'\vec e_z$ је у $\vec r'' = r_c'\vec e_{r_c'} - z'\vec e_z$ и има наелектрисање -1. $F = \frac{1}{4\pi\varepsilon_0}\frac{-1}{|\vec r - \vec r''|}$

$$G(\vec r, \vec r') = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{|r_c\vec e_{r_c} + z\vec e_z - z'\vec e_z - r_c'\vec e_{r_c'}|} - \frac{1}{|r_c\vec e_{r_c} + z\vec e_z - r_c'\vec e_{r_c'} + z'\vec e_z|}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(z-z')^2 + r_c^2 + r_c'^2 - 2r_cr_c'\cos(\varphi-\varphi')}} - \frac{1}{\sqrt{(z+z')^2 + r_c^2 + r_c'^2 - 2r_cr_c'\cos(\varphi-\varphi')}}\right)$$

б) $$\phi(\vec r) = \underbrace{\int \rho(\vec r')G(\vec r, \vec r')\,dV'}_{=0} - \varepsilon_0\oint_{\partial V} dS'\,\phi(\vec r')\frac{\partial G(\vec r, \vec r')}{\partial n'} \qquad \vec n' = -\vec e_z;\ \frac{\partial G}{\partial n'} = -\frac{\partial G}{\partial z'}$$

$$= \varepsilon_0\oint dS'\,\phi(\vec r')\frac{\partial G(\vec r, \vec r')}{\partial z'} = \varepsilon_0\int_0^a\int_0^{2\pi} r_c'dr_c'd\varphi'\,V_0\left.\frac{\partial G}{\partial z'}\right|_{z'=0}$$

$$= \cancel{\varepsilon_0}V_0\int_0^a r_c'dr_c'\int_0^{2\pi} d\varphi'\frac{1}{4\pi\cancel{\varepsilon_0}}\left(-\frac{1}{2}\right)\frac{2(-z)-2z}{(z^2+r_c^2+r_c'^2-2r_cr_c'\cos(\varphi-\varphi'))^{3/2}} = \boxed{\frac{V_0 z}{2\pi}\int_0^a r_c'dr_c'\int_0^{2\pi} d\varphi'\frac{1}{(z^2+r_c^2+r_c'^2-2r_cr_c'\cos(\varphi-\varphi'))^{3/2}}}$$

в) $$\phi(z\vec e_z) = \frac{V_0 z}{\cancel{2\pi}}\int_0^a r_c'dr_c'\int_0^{2\pi} d\varphi'\frac{1}{(z^2+r_c'^2)^{3/2}} = V_0 z\int_0^a\frac{d(z^2+r_c'^2)}{(z^2+r_c'^2)^{3/2}}\frac{1}{2} = \frac{V_0 z}{2}(-2)\left.\frac{1}{\sqrt{z^2+r_c'^2}}\right|_0^a$$

$$\phi(z\vec e_z) = V_0 z\left(\frac{1}{|z|} - \frac{1}{\sqrt{z^2+a^2}}\right) = \boxed{V_0\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right)} \leftarrow z>0$$

г) $$\phi(\vec r) = \frac{V_0 z}{2\pi}\int_0^a r_c'dr_c'\int_0^{2\pi} d\varphi'\frac{1}{(z^2+r_c^2)^{3/2}\left(1 + \frac{r_c'^2 - 2r_cr_c'\cos(\varphi-\varphi')}{z^2+r_c^2}\right)^{3/2}} \approx \frac{V_0 z}{2\pi}\int_0^a r_c'dr_c'\int_0^{2\pi} d\varphi'\frac{1}{(z^2+r_c^2)^{3/2}}\left[1 - \frac{3}{2}\frac{r_c'^2 - 2r_cr_c'\cos(\varphi-\varphi')}{z^2+r_c^2}\right.$$
$$\left. + \binom{-3/2}{2}\frac{(r_c'^2 - 2r_cr_c'\cos(\varphi-\varphi'))^2}{(z^2+r_c^2)^2} + \dots\right]$$

$$\binom{-3/2}{2} = \frac{(-3/2)(-5/2)}{2} = \frac{15}{8}$$

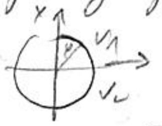
$$\phi(\vec r) = \frac{V_0 z}{2\pi}\int_0^a r_c'dr_c'\int_0^{2\pi}\frac{d\varphi'}{(z^2+r_c^2)^{3/2}}\left[1 - \frac{3}{2}\frac{r_c'^2 - \cancelto{0}{2r_cr_c'\cos(\varphi-\varphi')}}{z^2+r_c^2} + \frac{15}{8}\frac{r_c'^4 + 4r_c^2r_c'^2\overset{1/2}{\cos^2(\varphi-\varphi')} - \cancelto{0}{4r_c'^3r_c\cos(\varphi-\varphi')}}{(z^2+r_c^2)^2}\right]$$

$$= \frac{V_0 z}{(z^2+r_c^2)^{3/2}}\left(\frac{a^2}{2} - \frac{3}{2}\frac{\frac{a^4}{4}}{z^2+r_c^2} + \frac{15}{8}\frac{\frac{a^6}{6} + 4r_c^2\frac{a^4}{4}\frac{1}{2}}{(z^2+r_c^2)^2}\right) = \boxed{\frac{V_0a^2}{2}\frac{z}{(z^2+a^2)^{3/2}}\left(1 - \frac{3}{4}\frac{a^2}{z^2+r_c^2} + \frac{5}{8}\frac{a^4+3a^2r_c^2}{(r_c^2+a^2)^2}\right)}$$

а) Наћи Гринову ф-ју за дводимензионални потенцијал који је дат на површини цилиндра полупречника R, а у унутрашњости је дат са $\phi(r,\varphi)=\int_0^{2\pi}\phi(R,\varphi')\frac{(R^2-r^2)d\varphi'}{R^2+r^2-2Rr\cos(\varphi-\varphi')}\frac{1}{2\pi}$

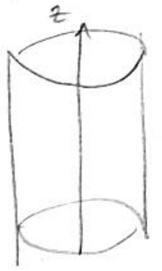
б) Две половине дуж проводног цилиндра полупречника R, налазе се на константном потенцијалу V_1 и V_2. Показати да је потенцијал у унутрашњости цилиндра:

$$\phi(r,\varphi)=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{\pi}\operatorname{arctg}\left(\frac{2Rr}{R^2-r^2}\cos\varphi\right)$$



в) Одредити индуковану површинску густину наелектрисања на цилиндру.

а)



Тражимо Г.Ф. која не зависи од z-осе, јер је и потенцијал инваријантан на транслације дуж z-осе!

$\boxed{2d}$ $\Delta G(\vec r_c,\vec r_c')=-\frac{1}{\varepsilon_0}\delta^{(2)}(\vec r_c-\vec r_c')$ $\qquad \vec r=\vec r_c+z\vec e_z$

$\Delta=\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$ $\qquad \delta^{(2)}(\vec r_c-\vec r_c')=\frac{1}{r_c}\delta(r_c-r_c')\delta(\varphi-\varphi')$

$$\left(\frac{1}{r_c}\frac{\partial}{\partial r_c}r_c\frac{\partial}{\partial r_c}+\frac{1}{r_c^2}\frac{\partial^2}{\partial\varphi^2}\right)G=-\frac{1}{\varepsilon_0}\frac{1}{r_c}\delta(r_c-r_c')\delta(\varphi-\varphi')$$

Аналогно Дирак-Гриновом идентитету у $\boxed{2d}$ важи: $\Delta\ln r_c=2\pi\delta(\vec r_c)$; $\vec r_c=r_c\vec e_{r_c}$

$$\Delta\ln\sqrt{r_c^2+a^2}=\frac{1}{r_c}\frac{\partial}{\partial r_c}r_c\frac{\partial}{\partial r_c}\ln\sqrt{r_c^2+a^2}=\frac{1}{r_c}\frac{\partial}{\partial r_c}\left(r_c\frac{r_c}{\sqrt{r_c^2+a^2}}\frac{1}{\sqrt{r_c^2+a^2}}\right)=\frac{1}{r_c}\frac{\partial}{\partial r_c}\left(\frac{r_c^2+a^2-a^2}{r_c^2+a^2}\right)$$

$$=\frac{1}{\cancel{r_c}}\frac{-a^2(-1)}{(r_c^2+a^2)^2}\cdot 2\cancel{r_c}=\frac{2a^2}{(r_c^2+a^2)^2}$$

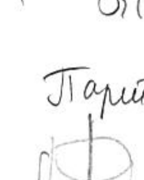
$$\int_0^\infty\int_0^{2\pi}r_c\,dr_c\,d\varphi\cdot\Delta\ln\sqrt{a^2+r_c^2}=\int_0^\infty\int_0^{2\pi}r_c\,dr_c\,d\varphi\frac{2a^2}{(r_c^2+a^2)^2}=4\pi a^2\int_0^\infty\frac{d(r_c^2+a^2)}{(r_c^2+a^2)^2}\frac{1}{2}$$

$$=2\pi a^2\frac{(-1)}{r_c^2+a^2}\Big|_0^\infty=2\pi\Rightarrow\boxed{\Delta\ln|\vec r_c-\vec r_c'|=2\pi\delta^{(2)}(\vec r_c-\vec r_c')}$$

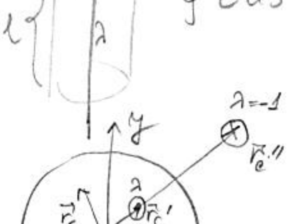
Добили смо партикуларно решење за Гринову ф-ју:

$$G(\vec r_c,\vec r_c')=-\frac{1}{2\pi\varepsilon_0}\ln|\vec r_c-\vec r_c'|+F(\vec r_c,\vec r_c')$$

Партикуларно решење представља потенцијал од линије са $\lambda=1$ у $\vec r_c'$!



$\oint\vec E\,d\vec S=\frac{1}{\varepsilon_0}\lambda l$; $E\cdot 2r\pi\cancel{l}=\frac{\lambda\cancel{l}}{\varepsilon_0}\Rightarrow E=\frac{\lambda}{2\pi\varepsilon_0 r}=-\frac{\partial V}{\partial r}\Rightarrow\boxed{V=-\frac{\lambda}{2\pi\varepsilon_0}\ln r}$



$$G(\vec r_c,\vec r_c')=-\frac{1}{2\pi\varepsilon_0}\ln\left(r_c^2+r_c'^2-2r_cr_c'\cos(\varphi-\varphi')\right)^{1/2}+F(\vec r_c,\vec r_c')$$

$$F(\vec r_c,\vec r_c')=-\frac{(-1)}{2\pi\varepsilon_0}\ln\left(r_c^2+r_c''^2-2r_cr_c''\cos(\varphi-\varphi')\right)^{1/2}+\underbrace{C(r_c',\varphi')}_{\text{не зависи од }\vec r_c}$$

$\Delta F(\vec r_c,\vec r_c')=0$ (први члан је лик ван области, други не зависи од $\vec r_c$)

$C(r_c',\varphi')$ одређујемо из услова $G(\vec r_c=R\vec e_{r_c},\vec r_c')=G(\vec r_c,\vec r_c'=R\vec e_{r_c'})=0$

Г.Ф. је симетрична.

$r_c''=\frac{R^2}{r_c'}$ – инверзија у односу на круг!

$G(r_c, r_c'=Re_{r_c'}) = -\frac{1}{4\pi\varepsilon_0}\ln(r_c^2+R^2-2r_cR\cos(\varphi-\varphi')) + \frac{1}{4\pi\varepsilon_0}\ln(r_c^2+R^2-2r_cR\cos(\varphi-\varphi')) + C(R,\varphi')=0$

$\Rightarrow \boxed{C(R,\varphi')=0}$

$G(\vec r_c = R\vec e_{r_c}, \vec r_c') = -\frac{1}{4\pi\varepsilon_0}\ln(R^2+r_c'^2-2Rr_c'\cos(\varphi-\varphi')) + \frac{1}{4\pi\varepsilon_0}\ln\left(R^2+\frac{R^4}{r_c'^2}-2R\frac{R^2}{r_c'}\cos(\varphi-\varphi')\right)+C(r_c',\varphi')=0$

$\Rightarrow C(r_c',\varphi') = \frac{1}{4\pi\varepsilon_0}\ln\left(\frac{R^2+r_c'^2-2Rr_c'\cos(\varphi-\varphi')}{R^2(r_c'^2+R^2-2r_c'R\cos(\varphi-\varphi'))}r_c'^2\right) = \frac{1}{4\pi\varepsilon_0}\ln\left(\frac{r_c'^2}{R^2}\right) = \boxed{\frac{1}{2\pi\varepsilon_0}\ln\frac{r_c'}{R}}$; $C(R,\varphi')=0$ задовољено!

$\Rightarrow G(\vec r_c,\vec r_c') = \frac{1}{4\pi\varepsilon_0}\ln\left(\frac{r_c^2r_c'^2+R^4-2r_cr_c'R^2\cos(\varphi-\varphi')}{\cancel{r_c'^2}(r_c^2+r_c'^2-2r_cr_c'\cos(\varphi-\varphi'))}\cdot\frac{\cancel{r_c'^2}}{R^2}\right) = \boxed{\frac{1}{4\pi\varepsilon_0}\ln\left[\frac{R^4+r_c^2r_c'^2-2R^2r_cr_c'\cos(\varphi-\varphi')}{R^2(r_c^2+r_c'^2-2r_cr_c'\cos(\varphi-\varphi'))}\right]}$

$\Phi(r,\varphi) = \int \underbrace{\rho(\vec r_c')}_{0} G(\vec r_c,\vec r_c')\underbrace{dv'}_{r_c'dr_c'd\varphi'} - \varepsilon_0\int \underbrace{ds'}_{Rd\varphi'}\Phi(\vec r_c')\frac{\partial G(\vec r_c,\vec r_c')}{\partial n'} = \boxed{-\varepsilon_0\int_0^{2\pi}Rd\varphi'\frac{\partial G}{\partial r_c'}\Big|_R\cdot\Phi(R\vec e_{r_c'})}$; $\vec n' = \vec e_{r_c'}$

$\frac{\partial G}{\partial r_c'}\Big|_R = \frac{1}{4\pi\varepsilon_0}\left[\frac{2Rr_c^2-2R^2r_c\cos(\varphi-\varphi')}{R^4+r_c^2R^2-2R^3r_c\cos(\varphi-\varphi')} - \frac{2R-2r_c\cos(\varphi-\varphi')}{r_c^2+R^2-2r_cR\cos(\varphi-\varphi')}\right] = \boxed{\frac{1}{2\pi\varepsilon_0}\frac{\frac{r_c^2}{R}-R}{r_c^2+R^2-2r_cR\cos(\varphi-\varphi')}}$

$\Phi(r,\varphi) = +\varepsilon_0 R\frac{R^2-r_c^2}{R}\frac{1}{2\pi\varepsilon_0}\int_0^{2\pi}d\varphi'\,\Phi(R\vec e_{r_c'})\frac{1}{r_c^2+R^2-2r_cR\cos(\varphi'-\varphi)} = \boxed{\frac{1}{2\pi}\int_0^{2\pi}\Phi(R,\varphi')\frac{(R^2-r_c^2)d\varphi'}{R^2+r_c^2-2Rr_c\cos(\varphi-\varphi')}}$

б) $\Phi(R\vec e_{r_c'}) = \begin{cases} V_1, & \varphi'\in(-\frac{\pi}{2},\frac{\pi}{2}) \\ V_2, & \varphi'\in(\frac{\pi}{2},\frac{3\pi}{2})\end{cases}$

$\Phi(r,\varphi) = \frac{V_1}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{(R^2-r_c^2)d\varphi'}{R^2+r_c^2-2Rr_c\cos(\varphi'-\varphi)} + \frac{V_2}{2\pi}\int_{\pi/2}^{3\pi/2}\frac{(R^2-r_c^2)d\varphi'}{R^2+r_c^2-2Rr_c\cos(\varphi'-\varphi)}$, $\boxed{\varphi'-\varphi = x}$

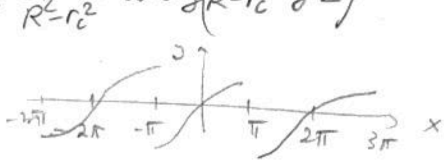
$= \frac{V_1}{2\pi}\int_{-\pi/2-\varphi}^{\pi/2-\varphi}\frac{(R^2-r_c^2)dx}{R^2+r_c^2-2Rr_c\cos x} + \frac{V_2}{2\pi}\int_{\frac{\pi}{2}-x}^{\frac{3\pi}{2}-x}\frac{(R^2-r_c^2)dx}{R^2+r_c^2-2Rr_c\cos x}$

$J = \int\frac{dx}{R^2+r_c^2-2Rr_c\cos x}$, смена $tg\frac{x}{2}=t$; $x = 2\,arctg\,t$, за $x\in(-\pi,\pi)$; $dx = \frac{2}{1+t^2}dt$

$t^2 = tg^2\frac{x}{2} = \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{1-\cos x}{1+\cos x} \to \cos x = \frac{1-t^2}{1+t^2}$

$J = \int\frac{\frac{2}{1+t^2}dt}{R^2+r_c^2-2Rr_c\frac{1-t^2}{1+t^2}} = 2\int\frac{dt}{(R-r_c)^2+t^2(R+r_c)^2} = \frac{2}{(R-r_c)^2}\int\frac{dt}{1+\left(t\frac{R+r_c}{R-r_c}\right)^2} = \frac{2}{(R-r_c)^2}\frac{R-r_c}{R+r_c}arctg\left(\frac{R+r_c}{R-r_c}tg\frac{x}{2}\right)+C$

$J = \frac{2}{R^2-r_c^2}arctg\left(\frac{R+r_c}{R-r_c}tg\frac{x}{2}\right)+const. \to$ није непрекидна фја за $x\in R$.



1° $\boxed{\varphi\in(-\frac{\pi}{2},\frac{\pi}{2})}$ $\quad -\frac{\pi}{2}-\varphi\in(-\pi,0)$; $\frac{\pi}{2}-\varphi\in(0,\pi)$; $\frac{3\pi}{2}-\varphi\in(\pi,2\pi)$

на интервалу $(-\frac{\pi}{2}-\varphi,\frac{\pi}{2}-\varphi)$ нема дисконтинуитета, а на интервалу $(\frac{\pi}{2}-\varphi,\frac{3\pi}{2}-\varphi)$ дисконт. у $\boxed{\pi}$!

$\Rightarrow \Phi(r,\varphi) = \frac{V_1}{2\pi}(R^2-r_c^2)\frac{2}{R^2-r_c^2}arctg\left(\frac{R+r_c}{R-r_c}tg\frac{x}{2}\right)\Big|_{-\pi/2-\varphi}^{\pi/2-\varphi} + \frac{V_2}{2\pi}(R^2-r_c^2)\frac{2}{R^2-r_c^2}arctg\left(\frac{R+r_c}{R-r_c}tg\frac{x}{2}\right)\Big|_{\frac{\pi}{2}-\varphi}^{\pi} + \frac{V_2}{2\pi}(R^2-r_c^2)\frac{2}{R^2-r_c^2}arctg\left(\frac{R+r_c}{R-r_c}tg\frac{x}{2}\right)\Big|_{\pi}^{\frac{3\pi}{2}-\varphi}$

$\Phi(r,\varphi) = \frac{V_1}{\pi}\left[arctg\left(\frac{R+r_c}{R-r_c}tg\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\right) - arctg\left(\frac{R+r_c}{R-r_c}tg\left(-\frac{\pi}{4}-\frac{\varphi}{2}\right)\right)\right] + \frac{V_2}{\pi}\left[arctg\left(\frac{R+r_c}{R-r_c}\underbrace{tg\left(\frac{3\pi}{4}-\frac{\varphi}{2}\right)}_{tg\left(-\frac{\pi}{4}-\frac{\varphi}{2}\right)}\right) - arctg\left(\frac{R+r_c}{R-r_c}tg\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\right)\right] + V_2$

$$\Phi(r,\varphi)=V_2+\frac{V_1-V_2}{\pi}\left[\operatorname{arctg}\left(\frac{R+r_c}{R-r_c}\operatorname{tg}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\right)+\operatorname{arctg}\left(\frac{R+r_c}{R-r_c}\operatorname{tg}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\right)\right]$$

$$=V_2+\frac{V_1-V_2}{\pi}\left[\underbrace{\operatorname{arctg}\left(\frac{R+r_c}{R-r_c}\operatorname{tg}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\right)}_{>0}+\underbrace{\operatorname{arctg}\left(\frac{R+r_c}{R-r_c}\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\right)}_{>0}\right]$$

$$\boxed{\operatorname{arctg}x+\operatorname{arctg}y=\operatorname{arctg}\frac{x+y}{1-xy}\quad\Big|\bmod\pi}\qquad\boxed{\frac{\pi}{4}-\frac{\varphi}{2}\in\left(0,\frac{\pi}{2}\right)}$$

$$\Phi(r,\varphi)=V_2+\frac{V_1-V_2}{\pi}\left[\operatorname{arctg}\underbrace{\left[\frac{\frac{R+r_c}{R-r_c}\left(\operatorname{tg}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)+\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\right)}{1-\left(\frac{R+r_c}{R-r_c}\right)^2}\right]}_{<0}+\pi\right]\;;\;\operatorname{tg}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)=\frac{1-\operatorname{tg}\frac{\varphi}{2}}{1+\operatorname{tg}\frac{\varphi}{2}}\;;\;\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)=\frac{1+\operatorname{tg}\frac{\varphi}{2}}{1-\operatorname{tg}\frac{\varphi}{2}}$$

$$\Phi(r,\varphi)=V_2+\frac{V_1-V_2}{\pi}\left[\operatorname{arctg}\left[\frac{R+r_c}{R-r_c}\cdot\frac{1}{1-\left(\frac{R+r_c}{R-r_c}\right)^2}\cdot\frac{\left(1-\operatorname{tg}\frac{\varphi}{2}\right)^2+\left(1+\operatorname{tg}\frac{\varphi}{2}\right)^2}{1-\operatorname{tg}^2\frac{\varphi}{2}}\right]+\pi\right]=$$

$$=V_2+\frac{V_1-V_2}{\pi}\left[\pi-\operatorname{arctg}\left(\frac{R^2-r_c^2}{4Rr_c}\,\frac{2\left(1+\operatorname{tg}^2\frac{\varphi}{2}\right)}{1-\operatorname{tg}^2\frac{\varphi}{2}}\right)\right]=V_2+\frac{V_1-V_2}{\pi}\left[\pi-\operatorname{arctg}\underbrace{\left(\frac{R^2-r_c^2}{2Rr_c\cos\varphi}\right)}_{>0}\right]$$

$$\boxed{\operatorname{arctg}x+\operatorname{arctg}\frac{1}{x}=\frac{\pi}{2}\quad\text{за }x>0}$$

$$\Phi(r,\varphi)=V_2+\frac{V_1-V_2}{\pi}\left[\pi-\left(\frac{\pi}{2}-\operatorname{arctg}\frac{2Rr_c\cos\varphi}{R^2-r_c^2}\right)\right]=V_2+\frac{V_1-V_2}{2}+\frac{V_1-V_2}{\pi}\operatorname{arctg}\left(\frac{2Rr_c\cos\varphi}{R^2-r_c^2}\right)$$

$$\boxed{\Phi(r,\varphi)=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{\pi}\operatorname{arctg}\left(\frac{2Rr_c\cos\varphi}{R^2-r_c^2}\right)}$$

2° $\boxed{\varphi\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)}$: $-\frac{\pi}{2}-\varphi\in(-2\pi,-\pi)$; $\frac{\pi}{2}-\varphi\in(-\pi,0)$; $\frac{3\pi}{2}-\varphi\in(0,\pi)$

на интервалу $\left(-\frac{\pi}{2}-\varphi,\frac{\pi}{2}-\varphi\right)$ дисконтинуитет у $\boxed{-\pi}$, на интервалу $\left(\frac{\pi}{2}-\varphi,\frac{3\pi}{2}-\varphi\right)$ нема дисконтинуитета

$$\Phi(r,\varphi)=V_1+\frac{V_1-V_2}{\pi}\left[\underbrace{\operatorname{arctg}\left(\frac{R+r_c}{R-r_c}\operatorname{tg}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\right)}_{<0}+\underbrace{\operatorname{arctg}\left(\frac{R+r_c}{R-r_c}\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\right)}_{<0}\right]=V_1+\frac{V_1-V_2}{\pi}\left(\operatorname{arctg}\underbrace{\left(\frac{\frac{R+r_c}{R-r_c}\left(\overset{<0}{\operatorname{tg}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)}+\overset{<0}{\operatorname{ctg}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)}\right)}{1-\left(\frac{R+r_c}{R-r_c}\right)^2}\right)}_{>0}-\pi\right)$$

$$=V_1+\frac{V_1-V_2}{\pi}\left[-\pi-\operatorname{arctg}\underbrace{\left(\frac{R^2-r_c^2}{2Rr_c}\,\frac{1+\operatorname{tg}^2\frac{\varphi}{2}}{1-\operatorname{tg}^2\frac{\varphi}{2}}\right)}_{<0}\right]=V_1+\frac{V_1-V_2}{\pi}\left[-\pi-\operatorname{arctg}\frac{R^2-r_c^2}{2Rr_c\underbrace{\cos\varphi}_{<0}}\right]$$

$$\boxed{\operatorname{arctg}x+\operatorname{arctg}\frac{1}{x}=-\frac{\pi}{2}\quad\text{за }x<0}$$

$$\Phi(r,\varphi)=V_1+\frac{V_1-V_2}{\pi}\left[-\pi-\left(-\frac{\pi}{2}-\operatorname{arctg}\left(\frac{2Rr_c\cos\varphi}{R^2-r_c^2}\right)\right)\right]=V_1+\frac{V_2-V_1}{2}+\frac{V_1-V_2}{\pi}\operatorname{arctg}\left(\frac{2Rr_c\cos\varphi}{R^2-r_c^2}\right)$$

$$\boxed{\Phi(r,\varphi)=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{\pi}\operatorname{arctg}\left(\frac{2Rr_c\cos\varphi}{R^2-r_c^2}\right)}$$