Решавање Лапласове једначине у Декартовим координатама

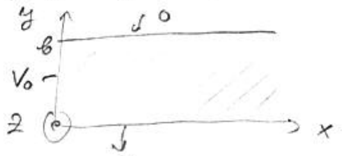
$\Delta V = 0, \rho=0 \quad V = V(x,y,z) = P(x)\,Q(y)\,R(z)$ ← раздвајање променљивих

$$\frac{d^2P(x)}{dx^2}Q(y)R(z) + P(x)\frac{d^2Q(y)}{dy^2}R(z) + P(x)Q(y)\frac{d^2R(z)}{dz^2} = 0 \;\Big/\; \frac{1}{PQR}$$

$$\underbrace{\frac{1}{P(x)}\frac{d^2P}{dx^2}}_{C_1} + \underbrace{\frac{1}{Q(y)}\frac{d^2Q}{dy^2}}_{C_2} + \underbrace{\frac{1}{R(z)}\frac{d^2R}{dz^2}}_{C_3} = 0 \Rightarrow C_1 + C_2 + C_3 = 0$$

знак константи одређује тип решења.

1. Одредити потенцијал електричног поља у области $x \geq 0$ која је ограничена равнима $x=0$, $y=0$, $y=b$. Раван $x=0$ је на константном потенцијалу V_0, док су друге две равни на нултом потенцијалу. Унутар ове области нема наелектрис



V не зависи од z-координате због трансл. симетрије

$V = P(x)\,Q(y) \rightarrow \quad \Delta V = 0 \quad \underbrace{\frac{1}{P}\frac{d^2P}{dx^2}}_{C_1 = C^2} + \underbrace{\frac{1}{Q}\frac{d^2Q}{dy^2}}_{C_2 = -C^2} = 0$

решење дуж y-правца мора да буде хармонијско $\Rightarrow C_2 = -C^2 < 0$

$Q(y) \sim \sin(Cy),\ \cos(Cy)$

дуж x-правца решења су експоненцијална $P(x) \sim e^{Cx}, e^{-Cx}$

$Q(y) = A\cdot \sin Cy + B\cos Cy \qquad Q(y=0) = Q(y=b) = 0 \Rightarrow B = 0,\ C\cdot b = m\pi,\ m\in\mathbb{N}$

$\Rightarrow Q(y) \sim \sin\frac{m\pi}{b}y$

$$V(x,y) = \sum_{m=0}^{\infty}\left(A_m e^{-\frac{m\pi}{b}x} + B_m e^{\frac{m\pi}{b}x}\right)\cdot \sin\frac{m\pi}{b}y$$

$V\big|_{x\to\infty} \to 0$ па је $B_m = 0,\ \forall m$

$$V(x=0,y) = V_0 = \sum_{m=0}^{\infty} A_m \sin\frac{m\pi}{b}y$$

Ф-је $\{\sin\frac{m\pi}{a}x\}$ су ортогоналне на интервалу $[0,a]$ $\quad m\in\mathbb{N}$

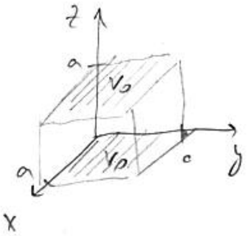
$$\boxed{\int_0^a \sin\frac{m\pi}{a}x\,\sin\frac{n\pi}{a}x\,dx = \frac{a}{2}\delta_{mn}}$$

$$\int_0^b V_0\cdot\sin\frac{n\pi}{b}y\,dy = \sum_{m=0}^{\infty} A_m\int_0^b \sin\frac{m\pi}{b}y\,\sin\frac{n\pi}{b}y\,dy = \sum_{m=0}^{\infty} A_m\frac{b}{2}\delta_{mn} = \frac{b}{2}A_n$$

$$A_n = \frac{2V_0}{b}\int_0^b \sin\frac{n\pi y}{b}\,dy = \frac{2V_0}{b}\frac{b}{n\pi}\left(-\cos\frac{n\pi y}{b}\right)\Big|_0^b = \frac{2V_0}{n\pi}\left(-(-1)^n + 1\right) \begin{cases} 0, & n = 2k,\ k\in\mathbb{N} \\ \frac{4V_0}{(2k+1)\pi}, & n = 2k+1,\ k\in\mathbb{N}_0 \end{cases}$$

$$\Rightarrow \boxed{V(x,y) = \frac{4V_0}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}e^{-\frac{(2k+1)\pi x}{b}}\cdot\sin\frac{(2k+1)\pi y}{b}}$$

2. Коцка проводних страница одређена је равнима $x=a$, $y=a$, $z=a$, $x=0$, $y=0$, $z=0$. Странице $z=a$ и $z=0$ су на потенцијалу V_0, док су остале стране коцке на нултом потенцијалу. Одредити потенцијал у унутрашњости коцке, ако у њој нема наелектрисања.



$$\Delta V = 0 \qquad V = P(x)\,Q(y)\,R(z)$$

$$\underbrace{\frac{1}{P}\frac{d^2P}{dx^2}}_{C_1} + \underbrace{\frac{1}{Q}\frac{d^2Q}{dy^2}}_{C_2} + \underbrace{\frac{1}{R}\frac{d^2R}{dz^2}}_{C_3} = 0$$

C_1 и $C_2 < 0$ харм. ім.
$C_3 > 0$

$$C_1 = -k_x^2 \qquad C_2 = -k_y^2 \qquad C_3 = k_x^2 + k_y^2 = k_z^2$$

$$P(x) = A_1 \sin k_x x + B_1 \cos k_x x \qquad P(x)\Big|_{\substack{x=0\\x=a}} = 0 \Rightarrow P(x) = A_1 \sin\frac{n\pi}{a}x,\; k_x = \frac{n\pi}{a},\; n\in\mathbb{N}$$

слично $Q(y) = A_2 \sin\frac{m\pi}{a}y$, $k_y = \frac{m\pi}{a}$, $m\in\mathbb{N}$

$$k_z^2 = \sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{a}\right)^2}^{\,2} = \frac{\pi}{a}\sqrt{n^2+m^2}$$

$$\frac{d^2R}{dz^2} - k_z^2 R = 0 \quad\Rightarrow\quad R(z) = A_3\,\mathrm{sh}\,k_z z + B_3\,\mathrm{ch}\,k_z z$$

$$\Rightarrow V(x,y,z) = \sum_{n,m=1}^{\infty}\left[D_{1nm}\,\mathrm{sh}\left(\frac{\pi}{a}\sqrt{n^2+m^2}\,z\right) + D_{2nm}\,\mathrm{ch}\left(\frac{\pi}{a}\sqrt{n^2+m^2}\,z\right)\right]\sin\frac{n\pi}{a}x\,\sin\frac{m\pi}{a}y$$

$$V(x,y,0) = V_0 = \sum_{n,m=1}^{\infty} D_{2nm}\sin\frac{n\pi}{a}x\,\sin\frac{m\pi}{a}y \qquad \Big/ \int_0^a\!\!\int_0^a \sin\frac{n'\pi}{a}x\,dx\,\sin\frac{m'\pi}{a}y\,dy$$

$$V_0\,\frac{a}{n'\pi}\frac{a}{m'\pi}\left(1-(-1)^{n'}\right)\left(1-(-1)^{m'}\right) = \sum_{n,m=1}^{\infty} D_{2nm}\,\frac{a}{2}\frac{a}{2}\,\delta_{nn'}\delta_{mm'} = \frac{a^2}{4}D_{2n'm'}$$

$$\Rightarrow D_{2nm} = \frac{4V_0}{\pi^2 nm}\left(1-(-1)^n\right)\left(1-(-1)^m\right)$$

$$V(x,y,a) = V_0 = \sum_{n,m=1}^{\infty}\left(D_{1nm}\,\mathrm{sh}\,\pi\sqrt{n^2+m^2} + D_{2nm}\,\mathrm{ch}\,\pi\sqrt{n^2+m^2}\right)\sin\frac{n\pi}{a}x\,\sin\frac{m\pi}{a}y \qquad \Big/ \int_0^a\!\!\int_0^a \sin\frac{n'\pi}{a}x\,\sin\frac{m'\pi}{a}y\,dx\,dy$$

$$V_0\,\frac{a}{n'\pi}\frac{a}{m'\pi}\left(1-(-1)^{n'}\right)\left(1-(-1)^{m'}\right) = \frac{a^2}{4}\left(D_{1n'm'}\,\mathrm{sh}\,\pi\sqrt{n'^2+m'^2} + D_{2n'm'}\,\mathrm{ch}\,\pi\sqrt{n'^2+m'^2}\right)$$

$$D_{1nm}\,\mathrm{sh}\,\pi\sqrt{n^2+m^2} + D_{2nm}\,\mathrm{ch}\,\pi\sqrt{n^2+m^2} = \frac{4V_0}{mn\pi^2}\left(1-(-1)^n\right)\left(1-(-1)^m\right)$$

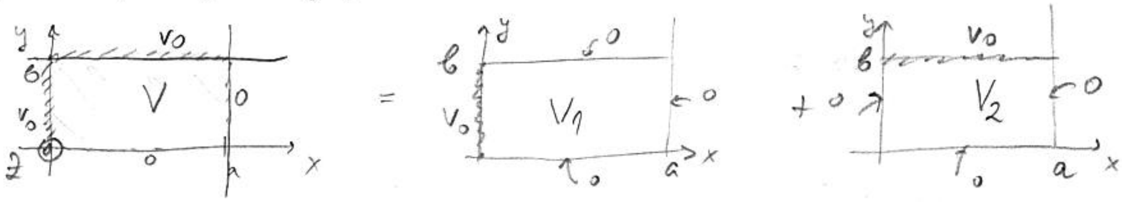
$$D_{1nm} = \frac{1}{\mathrm{sh}\,\pi\sqrt{n^2+m^2}}\,\frac{4V_0}{nm\pi^2}\left(1-(-1)^n\right)\left(1-(-1)^m\right)\left(1-\mathrm{ch}\,\pi\sqrt{n^2+m^2}\right)$$

$$V(x,y,z) = \frac{16V_0^2}{\pi^2}\sum_{n,m=0}^{\infty}\left[\frac{1-\mathrm{ch}\,\pi\sqrt{(2n+1)^2+(2m+1)^2}}{\mathrm{sh}\,\pi\sqrt{(2n+1)^2+(2m+1)^2}}\,\mathrm{sh}\left(\frac{\pi}{a}\sqrt{(2n+1)^2+(2m+1)^2}\,z\right) + \mathrm{ch}\left(\frac{\pi}{a}\sqrt{(2n+1)^2+(2m+1)^2}\,z\right)\right]\frac{\sin\frac{(2n+1)\pi x}{a}\sin\frac{(2m+1)\pi y}{a}}{(2n+1)(2m+1)}$$

$$= \boxed{\frac{16V_0^2}{\pi^2}\sum_{n,m=0}^{\infty}\left(\mathrm{sh}\left(\frac{\pi}{a}\sqrt{(2n+1)^2+(2m+1)^2}\,z\right) + \mathrm{sh}\left(\frac{\pi}{a}\sqrt{(2n+1)^2+(2m+1)^2}\,(a-z)\right)\right)\frac{\sin\frac{(2n+1)\pi x}{a}\sin\frac{(2m+1)\pi y}{b}}{\mathrm{sh}\,\pi\sqrt{(2n+1)^2+(2m+1)^2}\,(2n+1)(2m+1)}}$$

$$\left(\mathrm{sh}(x-y) = \mathrm{sh}x\cdot\mathrm{ch}y - \mathrm{sh}y\,\mathrm{ch}x\right)$$

3. Одредити електростатички потенцијал унутар бесконачног квадра одређеног равнима $x=0$, $x=a$, $y=0$, $y=b$. Потенцијал равни $x=0$ и $y=b$ је V_0, а потенцијал преосталих равни је нула. Унутар квадра нема наелектрисања.



Принцип суперпозиције.

$V = V_1 + V_2$ V_1 има ненулти гр. услов на $x=0$, а V_2 на $y=b$

$V_1 = P_1(x)\,Q_1(y) \rightarrow$ Q_1 - хармонијска решења P_1 - експоненцијална решења

$Q_1 \sim \sin k_y y$; $\cancel{\cos k_y y}$ 0 због гр. услова у $y=0$

$k_y b = n\pi$, $n \in \mathbb{N}$ гр. услов у $y=b$ $\Rightarrow \boxed{k_y = \frac{n\pi}{b}}$

$$V_1 = \sum_{n=1}^{\infty}\left(A_n \,\mathrm{sh}\frac{n\pi}{b}x + B_n \,\mathrm{ch}\frac{n\pi}{b}x\right)\sin\frac{n\pi}{b}y$$

$$V_1(x=0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{b}y = V_0 \quad \Big/ \int_0^b \sin\frac{m\pi}{b}y\,dy \Rightarrow V_0\frac{b}{m\pi}(1-(-1)^m) = \frac{b}{2}B_m$$

$$\boxed{B_m = \frac{2V_0}{m\pi}(1-(-1)^m)}$$

$$V_1(x=a) = 0 = \sum_{n=1}^{\infty}\left(A_n \,\mathrm{sh}\frac{n\pi a}{b} + B_n \,\mathrm{ch}\frac{n\pi a}{b}\right)\sin\frac{n\pi}{b}y \Rightarrow A_n = -B_n\frac{\mathrm{ch}\frac{n\pi a}{b}}{\mathrm{sh}\frac{n\pi a}{b}}$$

$$V_1(x,y) = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\left(\mathrm{ch}\frac{(2n+1)\pi}{b}x - \frac{\mathrm{sh}\frac{(2n+1)\pi}{b}x\cdot \mathrm{ch}\frac{(2n+1)\pi a}{b}}{\mathrm{sh}\frac{(2n+1)\pi a}{b}}\right)\frac{\sin\frac{(2n+1)\pi y}{b}}{2n+1}$$

$$\boxed{V_1(x,y) = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{\mathrm{sh}\left(\frac{(2n+1)\pi}{b}(a-x)\right)\cdot\sin\frac{(2n+1)\pi y}{b}}{(2n+1)\,\mathrm{sh}\frac{(2n+1)\pi a}{b}}}$$

$V_2 = P_2(x)\,Q_2(y) \rightarrow$ P_2 - хармонијска решења, Q_2 експоненцијална решења

$P_2 \sim \sin k_x x$, $\cancel{\cos k_x x}$ 0 због гр. услова у $x=0$

$k_x \cdot a = n\pi$, $n\in\mathbb{N}$ $\rightarrow \boxed{k_x = \frac{n\pi}{a}}$

$$V_2 = \sum_{n=1}^{\infty}\left(C_n \,\mathrm{sh}\frac{n\pi}{a}y + D_n \,\mathrm{ch}\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

$$V_2(y=0) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi}{a}x = 0 \Rightarrow D_n = 0 \quad \forall n\in\mathbb{N}$$

$$V_2(y=b) = \sum_{n=1}^{\infty} C_n \,\mathrm{sh}\frac{n\pi b}{a}\sin\frac{n\pi}{a}x = V_0 \quad \Big/ \int_0^a \sin\frac{m\pi}{a}x\,dx \Rightarrow V_0\frac{a}{m\pi}(1-(-1)^m) = \frac{a}{2}C_m \,\mathrm{sh}\left(\frac{m\pi b}{a}\right)$$

$$C_m = \frac{2V_0}{m\pi}(1-(-1)^m)\frac{1}{\mathrm{sh}\frac{m\pi b}{a}} \Rightarrow \boxed{V_2(x,y) = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{\mathrm{sh}\frac{(2n+1)\pi y}{a}\sin\frac{(2n+1)\pi}{a}x}{(2n+1)\cdot\mathrm{sh}\frac{(2n+1)\pi b}{a}}}$$

$$\boxed{V = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\left[\frac{\mathrm{sh}\frac{(2n+1)\pi(a-x)}{b}\cdot\sin\frac{(2n+1)\pi y}{b}}{(2n+1)\,\mathrm{sh}\frac{(2n+1)\pi a}{b}} + \frac{\mathrm{sh}\frac{(2n+1)\pi y}{a}\sin\frac{(2n+1)\pi}{a}x}{(2n+1)\,\mathrm{sh}\frac{(2n+1)\pi b}{a}}\right]}$$

Лапласова једначина у сферним координатама

Решава се јна $\Delta V = 0$, $V = V(r, \theta, \varphi)$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rV) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2} = 0, \quad V = \frac{R(r)}{r}P(\theta)Q(\varphi)$$

$$r^2\sin^2\theta\left[\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{r^2\sin\theta\, P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right] + \underbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}_{-m^2} = 0$$

$$\underbrace{\frac{r^2}{R}\frac{d^2R}{dr^2}}_{l(l+1)} + \underbrace{\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta}}_{-l(l+1)} = 0$$

$$\underbrace{\frac{d^2R}{dr^2} - \frac{l(l+1)}{r^2}R = 0}_{} \qquad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)P = 0$$

Ојлерова диф. јна решење $\sim r^k$ $\quad k\cdot(k-1) - l(l+1) = 0$, решења $k = l+1$, $k = -l$

$$\Rightarrow \quad R = Ar^{l+1} + \frac{B}{r^l}$$

Смена $x = \cos\theta$ $\quad x \in [-1, 1]$ у другу једначину она постаје:

$$\boxed{\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0}$$ Генералисана Лежандрова диф. јна

За $m = 0$ аксијална симетрија нема зависности од φ

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + l(l+1)P(x) = 0$$ Лежандрова диф. јна, решење Лежандрови полиноми

$$P_l(x) = \frac{1}{2^l \cdot l!}\frac{d^l}{dx^l}(x^2-1)^l, \quad l = 0, 1, \ldots$$

$\Rightarrow$ општи облик решења Лапласове јне за $m = 0$ је $\boxed{V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)}$

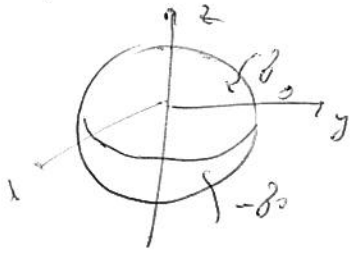
Решавање Лапласове јне у сферним координатама

$$\Delta V = 0$$

1° постоји аксијална симетрија $\quad V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$

2° нема симетрије $\quad V(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)Y_{lm}(\theta,\varphi)$

1. Сферна површ полупречника R састоји се од две полусфере које су равномерно наелектрисане површинским густинама σ_0 и $-\sigma_0$. Одредити потенцијал у целом пр.



$r<R$ област „<" ; $r>R$ област „>" ; $\Delta V = 0$

$$V_< = \sum_{l=0}^{\infty}\left(A_l r^l + \cancel{\frac{\tilde{A}_l}{r^{l+1}}}\right)P_l(\cos\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

дивергира у $r\to 0$

$$V_> = \sum_{l=0}^{\infty}\left(\cancel{\tilde{B}_l r^l} + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)$$

дивергира у $r\to\infty$

Гранични услови:

1° $V_< = V_> \big|_{r=R}$ $\qquad$ 2° $-\frac{\partial V_>}{\partial r} + \frac{\partial V_<}{\partial r}\Big|_{r=R} = \frac{\sigma}{\varepsilon_0} = \frac{1}{\varepsilon_0}\begin{cases}\sigma_0, & \theta\in(0,\frac{\pi}{2})\\ -\sigma_0, & \theta\in(\frac{\pi}{2},\pi)\end{cases}$

1° $\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta) \quad \Big/ \int_0^{\pi} P_m(\cos\theta)\sin\theta\, d\theta$

$$\sum_{l=0}^{\infty} A_l R^l \int_0^{\pi} P_l(\cos\theta)P_m(\cos\theta)\sin\theta\, d\theta = \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}\int_0^{\pi} P_l(\cos\theta)P_m(\cos\theta)\sin\theta\, d\theta$$

$$\sum_{l=0}^{\infty} A_l R^l \underbrace{\int_{-1}^{1} P_l(x)P_m(x)\,dx}_{\frac{2\delta_{lm}}{2l+1}} = \sum \frac{B_l}{R^{l+1}}\underbrace{\int_{-1}^{1} P_l(x)P_m(x)\,dx}_{\frac{2\delta_{lm}}{2l+1}}$$

$$\sum_{l=0}^{\infty} A_l R^l \frac{2\delta_{lm}}{2l+1} = \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}\frac{2\delta_{lm}}{2l+1} \Rightarrow A_m R^m \cancel{\frac{1}{2m+1}} = \cancel{\frac{1}{2m+1}}\frac{B_m}{R^{m+1}}$$

$$\Rightarrow \boxed{B_m = A_m R^{2m+1}}$$

2° $\sum_{l=0}^{\infty}\left[(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) + l A_l R^{l-1}P_l(\cos\theta)\right] = \frac{\sigma}{\varepsilon_0}$

$$\sum_{l=0}^{\infty}\left[(l+1)\frac{A_l R^{2l+1}}{R^{l+2}}P_l(\cos\theta) + l A_l R^{l-1}P_l(\cos\theta)\right] = \frac{\sigma}{\varepsilon_0}$$

$$\sum_{l=0}^{\infty} A_l(2l+1)R^{l-1}P_l(\cos\theta) = \frac{\sigma}{\varepsilon_0} \quad \Big/ \int_0^{\pi} P_m(\cos\theta)\sin\theta\, d\theta$$

$$\sum_{l=0}^{\infty} A_l(2l+1)R^{l-1}\int_{-1}^{1} P_l(x)P_m(x)\,dx = \frac{\sigma_0}{\varepsilon_0}\left[\int_0^{\pi/2} P_m(\cos\theta)\sin\theta\, d\theta - \int_{\pi/2}^{\pi} P_m(\cos\theta)\sin\theta\, d\theta\right]$$

$$\frac{2A_m\cancel{(2m+1)}R^{m-1}}{\cancel{2m+1}} = \frac{\sigma_0}{\varepsilon_0}\left[\int_0^1 P_m(x)\,dx - \int_{-1}^{0} P_m(x)\,dx\right] = \frac{\sigma_0}{\varepsilon_0}\left[\int_0^1 P_m(x)\,dx - \int_0^1 P_m(-x)\,dx\right]$$

$x = -x'$

$$= \frac{\sigma_0}{\varepsilon_0}\left(1-(-1)^m\right)\int_0^1 P_m(x)\,dx \Rightarrow A_m = \frac{\sigma_0\left(1-(-1)^m\right)}{2\varepsilon_0 R^{m-1}}\int_0^1 P_m(x)\,dx$$

$$B_m = \frac{\rho_0 (1-(-1)^m)}{2\varepsilon_0} R^{m+2} \int_0^1 P_m(x)\,dx$$

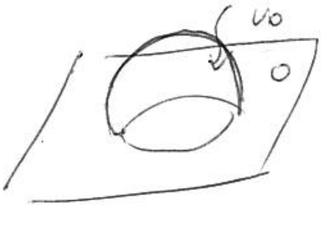
$$\int_0^1 P_m(x)\,dx = \begin{cases} \delta_{k0}, & \text{за } m=2k \\ \dfrac{(-1)^k (2k)!}{2^{2k+1} k!(k+1)!}, & \text{за } m=2k+1 \end{cases}$$

за $m=2k$ $\quad (1-(-1)^m)=0$ па парни свакако отпадају $\quad A_{2k}=B_{2k}=0$

$$A_{2k+1} = \frac{\rho_0}{\varepsilon_0}\frac{1}{R^{2k}} \int_0^1 P_{2k+1}(x)\,dx \qquad B_{2k+1} = \frac{\rho_0}{\varepsilon_0} R^{2k+3} \int_0^1 P_{2k+1}(x)\,dx$$

$$V = \begin{cases} \dfrac{\rho_0}{\varepsilon_0} \displaystyle\sum_{k=0}^{\infty} \frac{r^{2k+1}}{R^{2k}} \frac{(-1)^k (2k)!}{2^{2k+1} k!(k+1)!} P_{2k+1}(\cos\theta), & r \in (0, R) \\ \dfrac{\rho_0}{\varepsilon_0} \displaystyle\sum_{k=0}^{\infty} \frac{R^{2k+3}}{r^{2k+2}} \frac{(-1)^k (2k)!}{2^{2k+1} k!(k+1)!} P_{2k+1}(\cos\theta), & r \in [R, \infty) \end{cases}$$

2. Бесконачна раван има испупчење које има облик полусфере полупречника R. Потенцијал полусфере је V_0, док се раван одржава на нултом потенцијалу. Одредити потенцијал изнад равни, ако у том делу простора нема наелектрисања.



~~$r<R: V_<$~~ ; $\boxed{r>R: V_>}$ ово се тражи!

~~$V_< = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$~~ $\qquad V_> = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

$$V_>\left(\tfrac{\pi}{2}\right) = 0 = \underbrace{\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(0)}_{\text{не сме да зависи од } r} \Rightarrow P_{2k+1}(0)=0 \;; \quad P_{2k}(0) = \frac{(-1)^k (2k)!}{2^{2k}(k!)^2}$$

непарни не постоје због P_{2k+1}, парни $\boxed{B_{2k}=0}$

$$\Rightarrow V_> = \sum_{l=0}^{\infty} \frac{B_{2k+1}}{r^{2k+2}} P_{2k+1}(\cos\theta)$$

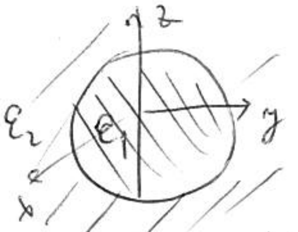
гр. услов: $\displaystyle\sum_{l=0}^{\infty} \frac{B_{2k+1}}{R^{2k+2}} P_{2k+1}(\cos\theta) = V_0 \quad \Big/ \int_0^{\pi/2} P_{2m+1}(\cos\theta)\sin\theta\,d\theta$

$$\sum_{l=0}^{\infty} \frac{B_{2k+1}}{R^{2k+2}} \underbrace{\int_0^1 P_{2k+1}(x) P_{2m+1}(x)\,dx}_{\frac{\delta_{km}}{2(2k+1)+1} = \frac{\delta_{km}}{4k+3}} = V_0 \underbrace{\int_0^1 P_{2m+1}(x)\,dx}_{\frac{(-1)^m (2m)!}{2^{2m+1} m!(m+1)!}}$$

$$\frac{B_{2m+1}}{R^{2m+2}(4m+3)} = V_0 \frac{(-1)^m (2m)!}{2^{2m+1} m!(m+1)!} \Rightarrow B_{2m+1} = V_0 \frac{(-1)^m (2m)!\,(4m+3)\,R^{2m+2}}{2^{2m+1} m!(m+1)!}$$

$$\boxed{V(r,\theta) = V_0 \sum_{m=0}^{\infty} \frac{(-1)^m (2m)!\,(4m+3)}{2^{2m+1} m!(m+1)!} \frac{R^{2m+2}}{r^{2m+2}} P_{2m+1}(\cos\theta)}$$

③ Хомогена сфера полупречника R непомичне диелектричне пропустљивости ε_1 налази се у бесконачном диелектрику диелектричне пропустљивости ε_2. На великим растојањима од сфере влада хомогено константно електрично поље E_0. Одредити потенцијал у целом простору и расподелу везаних наелектр.



У диелектрику $\mathrm{div}\,\vec{D} = \rho^{sl+ext}$ $\quad \vec{D} = -\varepsilon_0\varepsilon_r \nabla V$

$\Rightarrow \Delta V = 0$ за $\rho^{sl+ext} = 0$

$$\boxed{V_< = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)} \; ; \; V_> = \sum_{l=0}^{\infty}\left(\frac{B_l}{r^{l+1}} + C_l r^l\right) P_l(\cos\theta)$$

$$\vec{E}_> = -\mathrm{grad}\,V_> \Big|_{r\to\infty} = E_0 \vec{e}_z \qquad E_0\vec{e}_z = -\frac{\partial V_\infty}{\partial z}\vec{e}_z \Rightarrow V_\infty = -E_0 z + C$$

$$V_\infty = C - E_0 r\cos\theta = V_>\Big|_{r\to\infty} \Rightarrow C_0 = C \qquad C_1 = -E_0\,,\ C_l = 0\ \ l = 2,3,\dots$$

$$V_> = C - E_0 r\cos\theta + \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

гр. услови 1° $V_> = V_<\Big|_{r=R}$ $\qquad 2^\circ$ $D_{>n} - D_{<n} = \rho^{sl+ext} = 0$

$$1^\circ \quad \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = C - E_0 R\cos\theta + \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}} P_l(\cos\theta) \quad \Big/ \int_0^{\pi} P_m(\cos\theta)\sin\theta\, d\theta$$

$$\frac{2}{2m+1} A_m R^m = C\int_{-1}^{1} P_m(x)\,dx - E_0 R\int_{-1}^{1} P_1(x) P_m(x)\,dx + \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}\,\frac{2\delta_{lm}}{2m+1}$$

$$\frac{2}{2m+1} A_m R^m = 2C\,\delta_{m,0} - E_0 R\,\delta_{m,1}\frac{2}{3} + \frac{2}{2m+1}\,\frac{B_m}{R^{m+1}}$$

$m=0$: $\quad 2A_0 = 2C + \dfrac{2B_0}{R}$; $\quad m=1$ $\quad \dfrac{2}{3}A_1 R = -\dfrac{2}{3}E_0 R + \dfrac{2}{3}\dfrac{B_1}{R^2}$

$m>1$ $\quad B_m = A_m R^{2m+1}$

$\Rightarrow$ $B_0 = R(A_0 - C)$ $\;|\;$ $B_1 = (A_1 + E_0)R^3$

$$V_> = C - E_0 r\cos\theta + \frac{R(A_0 - C)}{r} + \frac{(A_1+E_0)R^3}{r^2}\cos\theta + \sum_{l=2}^{\infty}\frac{A_l R^{2l+1}}{r^{l+1}} P_l(\cos\theta)$$

$$\boxed{V_> = C - E_0 r\cos\theta - \frac{RC}{r} + \frac{E_0 R^3}{r^2}\cos\theta + \sum_{l=0}^{\infty}\frac{A_l R^{2l+1}}{r^{l+1}} P_l(\cos\theta)}$$

$$2^\circ \quad -\cancel{\varepsilon_0}\varepsilon_2\frac{\partial V_>}{\partial r} + \cancel{\varepsilon_0}\varepsilon_1\frac{\partial V_<}{\partial r}\Big|_{r=R} = 0 \Rightarrow \varepsilon_1\frac{\partial V_<}{\partial r} = \varepsilon_2\frac{\partial V_>}{\partial r}\Big|_{r=R}$$

$$\varepsilon_1\sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = \varepsilon_2\left(-E_0\cos\theta + \frac{C}{R} - 2E_0\cos\theta - \sum_{l=0}^{\infty}(l+1)\frac{A_l R^{2l+1}}{R^{l+2}} P_l(\cos\theta)\right)$$

$l=0$: $\quad \varepsilon_2\left(\dfrac{C}{R} - A_0 R^{-1}\right) = 0 \Rightarrow \boxed{A_0 = C}$; $\quad l=1$: $\varepsilon_1 A_1 = \varepsilon_2(-3E_0 - 2A_1) \Rightarrow \boxed{A_1 = \dfrac{-3E_0\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}}$

$l>1$ $\quad \varepsilon_1 l A_l R^{l-1} = -\varepsilon_2(l+1)A_l R^{l-1} \quad \Rightarrow \boxed{A_l = 0}$

$$V_< = C - \frac{3E_0\varepsilon_2}{\varepsilon_1+2\varepsilon_2} r\cos\theta \quad , r<R$$

$$V_> = C - E_0 r\cos\theta - \cancel{\frac{RC}{r}} + \frac{E_0 R^3}{r^2}\cos\theta + \cancel{\frac{CR}{r}} - \frac{3E_0\varepsilon_2}{\varepsilon_1+2\varepsilon_2}\frac{R^3}{r^2}\cos\theta \quad , r>R$$

$$V = \begin{cases} C - \dfrac{3E_0\varepsilon_2}{\varepsilon_1+2\varepsilon_2} r\cos\theta, & r<R \\ C - E_0 r\cos\theta + \dfrac{E_0(\varepsilon_1-\varepsilon_2)}{\varepsilon_1+2\varepsilon_2}\dfrac{R^3}{r^2}\cos\theta, & r>R \end{cases}$$

Везана наелектрисања

$$\mathrm{div}\,\vec{P} = -\rho^{vez} \qquad P_{2n} - P_{1n}\big|_R = -\sigma^{vez}$$

$$\vec{P} = \vec{D} - \varepsilon_0\vec{E} = (\varepsilon_0\varepsilon_r - \varepsilon_0)\vec{E} = -\varepsilon_0(\varepsilon_r - 1)\,\mathrm{grad}\,V$$

$$P_{>n} = -\varepsilon_0(\varepsilon_2-1)\frac{\partial V_>}{\partial r} \qquad P_{<n} = -\varepsilon_0(\varepsilon_1-1)\frac{\partial V_<}{\partial r}$$

$$\sigma^{vez} = P_{<n} - P_{>n}\big|_R = -\varepsilon_0(\varepsilon_1-1)\frac{\partial V_<}{\partial r}\Big|_R + \varepsilon_0(\varepsilon_2-1)\frac{\partial V_>}{\partial r}\Big|_R$$

$$= -\varepsilon_0(\varepsilon_1-1)\left(-\frac{3E_0\varepsilon_2}{\varepsilon_1+2\varepsilon_2}\right)\cos\theta + \varepsilon_0(\varepsilon_2-1)\left(-E_0\cos\theta - \frac{2E_0(\varepsilon_1-\varepsilon_2)}{\varepsilon_1+2\varepsilon_2}\cos\theta\right)$$

$$= \left(\frac{3E_0\varepsilon_0\varepsilon_2(\varepsilon_1-1)}{\varepsilon_1+2\varepsilon_2} - \varepsilon_0(\varepsilon_2-1)E_0\frac{3\varepsilon_1}{\varepsilon_1+2\varepsilon_2}\right)\cos\theta = \frac{3E_0\varepsilon_0(\cancel{\varepsilon_1\varepsilon_2} - \varepsilon_2 - \cancel{\varepsilon_1\varepsilon_2} + \varepsilon_1)}{\varepsilon_1+2\varepsilon_2}\cos\theta$$

$$\boxed{\sigma^{vez} = \frac{3E_0\varepsilon_0(\varepsilon_1-\varepsilon_2)}{\varepsilon_1+2\varepsilon_2}\cos\theta}$$

$$\mathrm{div}\,\vec{P}_< = \varepsilon_0(\varepsilon_1-1)\,\mathrm{div}\,\vec{E}_< = -\varepsilon_0(\varepsilon_1-1)\,\mathrm{div}\,\mathrm{grad}\,V_< = 0 \qquad \mathrm{div}\,\vec{P}_> = 0$$

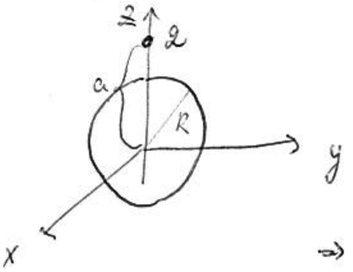
Нема запреминских везаних наелектрисања

(*) Проводна лопта полупречника R налази се у пољу тачкастог наелектрисања $\mathcal{Q}$, које је на растојању a $(a > R)$ од центра лопте. Систем се налази у хомогеном диелектрику, пропустљивости ε. Наћи потенцијал поља V и расподелу наелектрисања σ, индукованих на лопти, ако је задат:

а) потенцијал лопте V_0 (ако је $V=0$ за $r \to \infty$)

б) наелектрисање лопте Q.

Потенцијал представити у облику суме потенцијала неколико тачкастих наелектрисања $\mathcal{Q}$ и њихових ликова.



Тражимо потенцијал $V_<$ за $r<R$ и $V_>$ за $r>R$.

за $r<R$ $\quad \Delta V_< = 0$; $\operatorname{div}\vec{D} = \rho^{sl+ext} = 0 \quad \vec{D} = \varepsilon\vec{E} = -\varepsilon \operatorname{grad} V_<$

$\operatorname{div}\vec{D} = -\varepsilon \operatorname{div}\operatorname{grad} V = -\varepsilon \Delta V_< = 0$

$$\Rightarrow \boxed{\Delta V_< = 0} \qquad V_< = \sum_{l=0}^{\infty}\left(A_l r^l + \underbrace{\cancel{\frac{B_l}{r^{l+1}}}}_{0}\right) P_l(\cos\theta) = \boxed{\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)}$$

за $r>R$ $\quad \operatorname{div}\vec{D} = \rho^{sl+ext} = \mathcal{Q}\,\delta(\vec{r} - a\vec{e}_z) \qquad \vec{D} = \varepsilon\vec{E} = -\varepsilon \operatorname{grad} V_>$

$$-\varepsilon \Delta V_> = \mathcal{Q}\,\delta(\vec{r} - a\vec{e}_z) \Rightarrow \boxed{\Delta V_> = -\frac{\mathcal{Q}}{\varepsilon}\delta(\vec{r} - a\vec{e}_z)} \Rightarrow V_> = V_{>h} + V_{>p}$$

$$\Delta \frac{1}{|\vec{r} - \vec{r}'|} = -4\pi\,\delta(\vec{r} - \vec{r}') \quad \Rightarrow \quad V_{>p} = \frac{1}{4\pi\varepsilon}\frac{\mathcal{Q}}{|\vec{r} - a\vec{e}_z|} \qquad V_{>h} = \sum_{l=0}^{\infty}\left(\underbrace{\cancel{C_l}}_{0} r^l + \frac{D_l}{r^{l+1}}\right) P_l(\cos\theta)$$

$$\boxed{V_> = \frac{1}{4\pi\varepsilon}\frac{\mathcal{Q}}{|\vec{r} - a\vec{e}_z|} + \sum_{l=0}^{\infty}\frac{D_l}{r^{l+1}} P_l(\cos\theta)}$$

$V_<(r=R) = V_>(r=R) = V_0$ гранични услов

$$V_<(R) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0 \quad \Big/ \int_0^{\pi} P_m(\cos\theta)\sin\theta\, d\theta$$

$$\sum_{l=0}^{\infty} A_l R^l \int_{-1}^{1} P_l(x) P_m(x)\, dx = V_0 \int_{-1}^{1} P_m(x)\, dx$$

$$\sum_{l=0}^{\infty} A_l R^l \frac{\cancel{2}\delta_{lm}}{2l+1} = V_0 \frac{\cancel{2}\delta_{0m}}{2\cdot 0+1} \quad \Rightarrow \quad A_m R^m \frac{1}{2m+1} = V_0 \Rightarrow \boxed{A_m = \delta_{0m} V_0 \frac{2m+1}{R^m}}$$

$$\Rightarrow \boxed{V_<(r) = V_0}$$

за $r > R$ $\quad V_>(R) = V_0$

$$V_>(R) = \sum_{l=0}^{\infty}\frac{D_l}{R^{l+1}} P_l(\cos\theta) + \frac{1}{4\pi\varepsilon}\frac{\mathcal{Q}}{|R\vec{e}_r - a\vec{e}_z|} = V_0 \quad \Big/ \int_0^{\pi} P_m(\cos\theta)\sin\theta\, d\theta$$

$$\sum_{l=0}^{\infty}\frac{D_l}{R^{l+1}}\int_{-1}^{1} P_l(x) P_m(x)\, dx + \frac{\mathcal{Q}}{4\pi\varepsilon_0}\int_0^{\pi} P_m(\cos\theta)\sin\theta\, d\theta \frac{1}{|R\vec{e}_r - a\vec{e}_z|} = V_0 \int_0^{\pi} P_m(\cos\theta)\sin\theta\, d\theta$$

ово треба разложити по Лежандровим полиномима:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} \qquad \gamma = \measuredangle(\vec{r}, \vec{r}')$$

$= f(\cos\gamma)$ – може да се разложи по полиномима $P_l(\cos\gamma)$

(2)

$$\frac{1}{|\vec r-\vec r'|}=\sum_{l=0}^{\infty} C_l\, P_l(\cos\gamma)$$ C_l не зависи од γ! само од r и r'!

за $\gamma=0$ $$\frac{1}{|\vec r-\vec r'|}=\begin{cases}\dfrac{1}{r-r'}, & \text{за } r>r'\\[2mm] \dfrac{1}{r'-r}, & \text{за } r<r'\end{cases}$$ $\max\{r,r'\}=r_>$; $\min\{r,r'\}=r_<$

$$\frac{1}{|\vec r-\vec r'|}=\frac{1}{r_>-r_<}=\frac{1}{r_>\left(1-\frac{r_<}{r_>}\right)}=\frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l=\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}=\sum_{l=0}^{\infty}C_l P_l(\cos 0)=\sum_{l=0}^{\infty}C_l$$

$$\Rightarrow \boxed{C_l=\frac{r_<^l}{r_>^{l+1}}}$$

$$\Rightarrow \frac{1}{|\vec r-\vec r'|}=\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)$$

$$\sum_{l=0}^{\infty}\frac{D_l}{R^{l+1}}\int_{-1}^{1}P_l(x)P_m(x)\,dx+\frac{q}{4\pi\varepsilon}\sum_{l=0}\int_{-1}^{1}\frac{R^l}{a^{l+1}}P_l(x)P_m(x)\,dx=V_0\int_{-1}^{1}P_m(x)\,dx$$

$$\frac{D_m}{R^{m+1}}\frac{\cancel{2}}{2m+1}+\frac{q}{4\pi\varepsilon}\frac{R^m}{a^{m+1}}\frac{\cancel{2}}{2m+1}=V_0\,\cancel{2}\delta_{m,0}$$

$m=0$ $$\frac{D_0}{R}+\frac{q}{4\pi\varepsilon}\frac{1}{a}=V_0 \Rightarrow \boxed{D_0=V_0R-\frac{q}{4\pi\varepsilon}\frac{R}{a}}$$

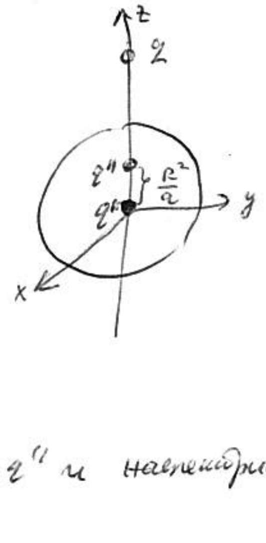
$m\neq0$ $$\frac{D_m}{R^{m+1}}\frac{\cancel{1}}{\cancel{2m+1}}=-\frac{q}{4\pi\varepsilon}\frac{R^m}{a^{m+1}}\frac{\cancel{1}}{\cancel{2m+1}} \Rightarrow \boxed{D_m=-\frac{q}{4\pi\varepsilon}\frac{R^{2m+1}}{a^{m+1}}}$$

$$V_>=\sum_{l=0}^{\infty}\frac{D_l}{r^{l+1}}P_l(\cos\theta)+\frac{1}{4\pi\varepsilon}\frac{q}{|\vec r-a\vec e_z|}=\frac{D_0}{r}+\sum_{m=1}^{\infty}\left(-\frac{q}{4\pi\varepsilon}\right)\frac{R^{2m+1}}{a^{m+1}}\frac{1}{r^{m+1}}P_m(\cos\theta)+\frac{1}{4\pi\varepsilon}\frac{q}{|\vec r-a\vec e_z|}$$

$$=\frac{V_0R}{r}-\underbrace{\frac{q}{4\pi\varepsilon}\sum_{m=0}^{\infty}\frac{R^{2m+1}}{a^{m+1}}\frac{1}{r^{m+1}}P_m(\cos\theta)}_{\text{треба написати као потенцијал неког лика}}+\frac{1}{4\pi\varepsilon}\frac{q}{|\vec r-a\vec e_z|}$$

$$\sum_{m=0}^{\infty}\frac{R^{2m+1}}{a^{m+1}}\frac{1}{r^{m+1}}P_m(\cos\theta)=R\sum_{m=0}^{\infty}\frac{1}{ar}\left(\frac{R^2}{ar}\right)^m P_m(\cos\theta)=\frac{R}{a}\sum_{m=0}^{\infty}\frac{\left(\frac{R^2}{a}\right)^m}{r^{m+1}}P_m(\cos\theta)$$

$$=\frac{R}{a}\frac{1}{\left|\frac{R^2}{a}\vec e_z-\vec r\right|}$$



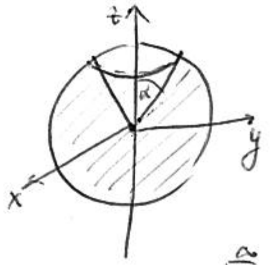
$$\boxed{V_>=\frac{1}{4\pi\varepsilon}\frac{4\pi\varepsilon V_0R}{|\vec r-0|}+\frac{1}{4\pi\varepsilon}\frac{-q\frac{R}{a}}{\left|\vec r-\frac{R^2}{a}\vec e_z\right|}+\frac{1}{4\pi\varepsilon}\frac{q}{|\vec r-a\vec e_z|}}$$

$q'=4\pi\varepsilon V_0R$; $q''=-q\frac{R}{a}$

$r'=0$ $\qquad r''=\frac{R^2}{a}$

Потенцијал ван кугле је сума потенцијала ликова q' и q'' и наелектрисања q.

*) Сфера полупречника R равномерно је наелектрисана густином наелектрисања σ, по јединици површине изузев сегмента који одговара углу $\theta=\alpha$. Одредити потенцијал у целом простору, као и поље у центру сфере.



Аксијална симетрија:

$$\Delta V = 0 \Rightarrow V_< = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \; ; \; V_> = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Гр. услов: $V_< = V_> \big|_{r=R}$

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) \Rightarrow \boxed{B_l = A_l R^{2l+1}} \quad \forall l$$

$$E_{>n} - E_{<n} = \frac{\sigma}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0} \cdot \begin{cases} 0, & \theta \in (0,\alpha) \\ 1, & \theta \in (\alpha,\pi) \end{cases} = \frac{\sigma}{\varepsilon_0}\eta(\theta-\alpha)$$

$$-\frac{\partial V_>}{\partial r} + \frac{\partial V_<}{\partial r}\Big|_R = \frac{\sigma}{\varepsilon_0}\eta(\theta-\alpha)$$

$$\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) + \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \frac{\sigma}{\varepsilon_0}\eta(\theta-\alpha) \Big/ \int_0^{\pi} P_m(\cos\theta)\sin\theta\, d\theta$$

$$\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}\int_{-1}^{1} P_l(x)P_m(x)\,dx = \frac{\sigma}{\varepsilon_0}\int_{\alpha}^{\pi} P_m(\cos\theta)\sin\theta\, d\theta$$

$$\sum_{l=0}^{\infty}\cancel{(2l+1)}A_l R^{l-1}\frac{2\delta_{l,m}}{\cancel{2l+1}} = \frac{\sigma}{\varepsilon_0}\int_{-1}^{\cos\alpha} P_m(x)\,dx \Rightarrow A_m = \frac{\sigma}{2\varepsilon_0 R^{m-1}}\underbrace{\int_{-1}^{\cos\alpha} P_m(x)\,dx}_{\text{Треба наћи преко } P_l}$$

особина (10): $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$

(12): $P'_{n+1}(x) - xP'_n(x) = (n+1)P_n(x)$

(10) диференцирамо: $(n+1)P'_{n+1}(x) - (2n+1)P_n(x) - (2n+1)xP'_n(x) + nP'_{n-1}(x) = 0$

из (12): $xP'_n(x) = P'_{n+1}(x) - (n+1)P_n(x)$

$$(n+1)P'_{n+1}(x) - (2n+1)P_n(x) - (2n+1)P'_{n+1}(x) + (2n+1)(n+1)P_n(x) + nP'_{n-1}(x) = 0$$

$$n(2n+1)P_n(x) = nP'_{n+1}(x) - nP'_{n-1}(x) \Rightarrow P_n(x) = \frac{1}{2n+1}\left(P'_{n+1}(x) - P'_{n-1}(x)\right)$$

За $n=0$ желимо да ова релација важи $\quad P_0(x) = 1 = \frac{1}{2\cdot 0+1}\Big(\underbrace{x'}_{1} - P'_{0-1}(x)\Big) \Rightarrow P'_{0-1} = P'_{-1} = 0$

$$m>0 \quad \int_{-1}^{\cos\alpha} P_m(x)\,dx = \int_{-1}^{\cos\alpha}\frac{1}{2m+1}\left(P'_{m+1}(x) - P'_{m-1}(x)\right)dx = \frac{1}{2m+1}\Big[P_{m+1}(x) - P_{m-1}(x)\Big]\Big|_{-1}^{\cos\alpha}$$

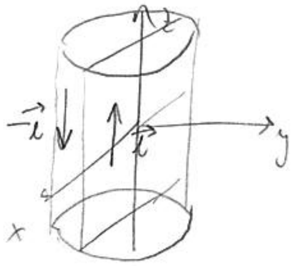
$$= \frac{1}{2m+1}\Big[P_{m+1}(\cos\alpha) - \underbrace{\cancel{P_{m+1}(-1)}}_{(-1)^{m+1}} - P_{m-1}(\cos\alpha) + \underbrace{\cancel{P_{m-1}(-1)}}_{(-1)^{m-1}}\Big] = \boxed{\frac{1}{2m+1}\Big[P_{m+1}(\cos\alpha) - P_{m-1}(\cos\alpha)\Big]}$$

$$\text{За } m=0 \quad \int_{-1}^{\cos\alpha} P_0(x)\,dx = \int_{-1}^{\cos\alpha} 1\cdot dx = \cos\alpha + 1 \Rightarrow \boxed{P_{-1}(x) \equiv -1}$$ ово дефинишемо

$$\boxed{\begin{aligned} V_< &= \frac{\sigma}{2\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left(P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right)\frac{r^l}{R^{l-1}}P_l(\cos\theta) \\ V_> &= \frac{\sigma}{2\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left(P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right)\frac{R^{l+2}}{r^{l+1}}P_l(\cos\theta) \end{aligned}}$$

Лапласова ј-на у цилиндричним координатама

По једној половини бесконачног цилиндра полупречника R паралелно његовој оси тече струја површинске густине $\vec{i}$, а по његовој другој половини тече струја $-\vec{i}$. Одредити енергију магнетног поља по јединици дужине цилиндра.



$$W = \int \frac{1}{2\mu_0} \vec{B}^2 d^3\vec{r} \quad \leftarrow \text{треба да нађемо } \vec{B}\,!$$

$$\vec{B} = \mathrm{rot}\,\vec{A} \qquad \Delta \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j}$$

„$<$" област $r<R$; „$>$" област $r>R$

решавамо $\Delta \vec{A} = 0$ у областима „$<$" и „$>$"

За векторски потенцијал претпостављамо правац паралелан струји

$$\Rightarrow \vec{A} = F(r)\,G(\varphi)\,\vec{e}_z \to \text{не зависи од } z$$

$$\Delta \vec{A} = \Delta A_z \cdot \vec{e}_z \; ; \; \Delta A_z = \frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r}\frac{\partial A_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \varphi^2} + \cancelto{0}{\frac{\partial^2 A_z}{\partial z^2}} = 0$$

$$F''(r)\cdot G(\varphi) + \frac{1}{r}F'(r)G(\varphi) + \frac{1}{r^2}F(r)G''(\varphi) = 0 \quad \Big/ \frac{r^2}{F(r)G(\varphi)}$$

$$\underbrace{\frac{r^2}{F(r)}F''(r) + \frac{r}{F(r)}F'(r)}_{m^2} + \underbrace{\frac{1}{G(\varphi)}G''(\varphi)}_{-m^2} = 0$$

Решење по φ – мора да буде периодично, са периодом 2π, $m \in \mathbb{N}$

$$G'' + m^2 G = 0 \Rightarrow \boxed{G \sim \sin m\varphi, \cos m\varphi}$$

$$\frac{r^2}{F}F'' + \frac{r}{F}F' = m^2 \Rightarrow F'' + \frac{1}{r}F' - \frac{m^2}{r^2}F = 0 \quad \text{Ојлерова ј-на} \quad F \sim r^\alpha$$

$$\alpha(\alpha - \cancel{1}) + \cancel{\alpha} - m^2 = 0 \Rightarrow \alpha = \pm m \quad \boxed{F \sim r^m, r^{-m}}$$

Случај $m=0$: $F'' + \frac{1}{r}F' = 0 \qquad \frac{dF'}{dr} = -\frac{1}{r}F' \Rightarrow \frac{dF'}{F'} = -\frac{dr}{r} \Big/ \int$

$$\ln F' = -\ln r + \mathrm{const} \quad F' = \frac{C}{r} \Rightarrow \boxed{F(r) = C\ln r + D}$$

$$G'' = 0 \quad \boxed{G(\varphi) = \cancelto{0}{A}\varphi + B}$$

нефизички да се G повећава са φ

$$A_< = \sum_{m=1}^{\infty}(A_m \sin m\varphi + B_m \cos m\varphi)\, r^m + \cancelto{0}{E_1}\ln r + F_1 \qquad r<R$$

дивергира у $r \to 0$

$$A_> = \sum_{m=1}^{\infty}(C_m \sin m\varphi + D_m \cos m\varphi)\frac{1}{r^m} + E_2 \ln r + F_2 \qquad r>R$$

Понашање $\vec{B}$ у бесконачности $\quad \oint \vec{B}\,d\vec{l} = \mu_0 I = 0 \quad B \sim \frac{1}{r^{\alpha+1}} \quad \alpha > 0$ јер $B \cdot 2r\pi \to 0$ за $r \to \infty$

$$\Rightarrow E_2 = 0$$

$$\Rightarrow \quad A_z(r,\varphi) = \begin{cases} \sum\limits_{m=0} (A_m \sin m\varphi + B_m \cos m\varphi)\, r^m, & r<R \\ \sum\limits_{m=0}^{\infty} (C_m \sin m\varphi + D_m \cos m\varphi)\, \frac{1}{r^m}, & r>R \end{cases}$$

Тражимо константе A_m, B_m, C_m, D_m

1° $A_< = A_> \big|_{r=R}$

$$\int_0^{2\pi} \sin m\varphi \sin n\varphi \, d\varphi = \pi \delta_{m,n} \qquad \int_0^{2\pi} \cos m\varphi \cos n\varphi \, d\varphi = \pi \delta_{n,m} \qquad \int_0^{2\pi} \cos m\varphi \sin n\varphi \, d\varphi = 0 \quad \underline{m>0}$$

$$\sum_{m=0}^{\infty} (A_m \sin m\varphi + B_m \cos m\varphi) R^m = \sum_{m=0}^{\infty} (C_m \sin m\varphi + D_m \cos m\varphi) \frac{1}{R^m} \qquad \Big/ \int_0^{2\pi} \sin n\varphi \, d\varphi \quad n\neq 0$$
$$\Big/ \int_0^{2\pi} \cos n\varphi \, d\varphi \quad n\neq 0$$

$$\Rightarrow A_m R^m = C_m \frac{1}{R^m} \Rightarrow \boxed{C_m = A_m R^{2m}}$$

$$\Rightarrow B_m R^m = \frac{D_m}{R^m} \rightarrow \boxed{D_m = B_m R^{2m}}$$

2° $\vec{n} \times (\vec{B}_> - \vec{B}_<) = \mu_0 i \vec{e}_z \begin{cases} 1, & \varphi \in (0,\pi) \\ -1, & \varphi \in (\pi, 2\pi) \end{cases}$

$$\vec{B} = \mathrm{rot}\vec{A} = \mathrm{rot}(A \vec{e}_z) = \underbrace{\frac{1}{r}\frac{\partial A}{\partial \varphi}\vec{e}_r}_{\text{ово не доприноси у гр. услову.}} - \frac{\partial A}{\partial r}\vec{e}_\varphi \qquad \vec{n} = \vec{e}_r$$

$$\vec{e}_r \times \left(-\frac{\partial A_>}{\partial r} + \frac{\partial A_<}{\partial r}\right)\vec{e}_\varphi \Big|_{r=R} = \mu_0 i \vec{e}_z \begin{cases} 1; & \varphi \in (0,\pi) \\ -1; & \varphi \in (\pi, 2\pi) \end{cases}$$

$$\Rightarrow \frac{\partial A_<}{\partial r} - \frac{\partial A_>}{\partial r}\Big|_{r=R} = \mu_0 i \begin{cases} 1; & \varphi \in (0,\pi) \\ -1; & \varphi \in (\pi, 2\pi) \end{cases}$$

$$\sum_{m=0}^{\infty} \left[m (A_m \sin m\varphi + B_m \cos m\varphi) R^{m-1} + m (C_m \sin m\varphi + D_m \cos m\varphi) \frac{1}{R^{m+1}} \right] = \mu_0 i \begin{cases} 1, & \varphi \in (0,\pi) \\ -1, & \varphi \in (\pi, 2\pi) \end{cases}$$

$$\Rightarrow \sum_{m=0}^{\infty} 2m (A_m \sin m\varphi + B_m \cos m\varphi) R^{m-1} = \mu_0 i \begin{cases} 1, & \varphi \in (0,\pi) \\ -1, & \varphi \in (\pi, 2\pi) \end{cases} \qquad \Big/ \int_0^{2\pi} \sin n\varphi \, d\varphi$$
$$\Big/ \int_0^{2\pi} \cos n\varphi \, d\varphi$$

$$\sum_{m=1}^{\infty} 2m A_m \pi \delta_{n,m} R^{m-1} = \mu_0 i \left(\int_0^{\pi} \sin n\varphi \, d\varphi - \int_{\pi}^{2\pi} \sin n\varphi \, d\varphi \right)$$

$$2n\pi A_n R^{n-1} = \mu_0 i \frac{1}{n}\left(-(-1)^n + 1 + 1 - (-1)^n\right) = 2\mu_0 i \left(1 - (-1)^n\right)$$

$$\Rightarrow n = 2k \Rightarrow A_{2k} = 0 \qquad n = 2k+1 \quad A_{2k+1} = \frac{4\mu_0 i}{2(2k+1)^2 \pi R^{2k}} \boxed{= \frac{2\mu_0 i}{(2k+1)^2 \pi R^{2k}}}$$

$$\sum_{m=1}^{\infty} 2m B_m \pi \delta_{n,m} R^{m-1} = \mu_0 i \left(\int_0^{\pi} \cos n\varphi \, d\varphi - \int_{\pi}^{2\pi} \cos n\varphi \, d\varphi \right) = 0 \Rightarrow B_n = 0 \quad n>0$$

за $n=0$ узмемо $B_0 = 0$ $\qquad C_{2k+1} = A_{2k+1} R^{4k+2} = \boxed{\frac{2\mu_0 i}{(2k+1)^2 \pi} R^{2k+2}}$

$$\vec{A}(r,\varphi) = \begin{cases} \sum\limits_{k=0}^{\infty} \frac{2\mu_0 i}{\pi(2k+1)^2} \frac{r^{2k+1}}{R^{2k}} \sin(2k+1)\varphi \, \vec{e}_z, & r<R \\ \sum\limits_{k=0}^{\infty} \frac{2\mu_0 i}{\pi(2k+1)^2} \frac{R^{2k+2}}{r^{2k+1}} \sin(2k+1)\varphi \, \vec{e}_z, & r>R \end{cases}$$

$$\vec{B}(r,\varphi) = \frac{2\mu_0 i}{\pi} \begin{cases} \sum\limits_{k=0}^{\infty} \left(\frac{r^{2k}}{R^{2k}(2k+1)} \cos(2k+1)\varphi \, \vec{e}_r - \frac{r^{2k}}{R^{2k}(2k+1)} \sin(2k+1)\varphi \, \vec{e}_\varphi \right), & r<R \\ \sum\limits_{k=0}^{\infty} \left(\frac{R^{2k+2}}{r^{2k+2}(2k+1)} \cos(2k+1)\varphi \, \vec{e}_\varphi + \frac{R^{2k+2}}{r^{2k+2}(2k+1)} \sin(2k+1)\varphi \, \vec{e}_\varphi \right), & r>R \end{cases}$$

$$dW=\frac{1}{2\mu_0}\left(\int_0^R\int_0^{2\pi}\vec{B}_<^2\, r\,dr\,d\varphi\,dz+\int_R^\infty\int_0^{2\pi}\vec{B}_>^2\, r\,dr\,d\varphi\,dz\right)$$

$$\frac{dW}{dz}=\frac{1}{2\mu_0}\left(\int_0^R\int_0^{2\pi}\vec{B}_<^2\, r\,dr\,d\varphi+\int_R^\infty\int_0^{2\pi}\vec{B}_>^2\, r\,dr\,d\varphi\right)$$

$$\frac{dW}{dz}=\frac{1}{2}\int_0^\pi \vec{A}_<(R)\vec{i}\,R\,d\varphi-\frac{1}{2}\int_\pi^{2\pi} A_<(R)\vec{i}\,R\,d\varphi$$
$$=\frac{2\mu_0 i^2}{2\pi}\sum_{k=0}^\infty\frac{R^2}{(2k+1)^2}\left(\int_0^\pi\sin((2k+1)\varphi)\,d\varphi-\int_\pi^{2\pi}\sin((2k+1)\varphi)\,d\varphi\right)$$
$$=\frac{4\mu_0 i^2R^2}{\pi}\sum_{k=0}^\infty\frac{1}{(2k+1)^3}=\boxed{\frac{7\mu_0 i^2R^2}{2\pi}\zeta(3)}$$

$$=\frac{1}{2\mu_0}\left(\frac{2\mu_0 i}{\pi}\right)^2\Bigg\{\sum_{k,l=0}^\infty\left[\int_0^R\int_0^{2\pi}\frac{r^{2k+2l}}{R^{2k+2l}(2k+1)(2l+1)}\Big(\underbrace{\cos((2k+1)\varphi)\cos((2l+1)\varphi)}_{\pi\delta_{k,l}}+\underbrace{\sin((2k+1)\varphi)\sin((2l+1)\varphi)}_{\pi\delta_{k,l}}\Big)r\,dr\,d\varphi\right]$$

$$+\sum_{k,l=0}^\infty\int_R^\infty\int_0^{2\pi}\frac{R^{2k+2+2l+2}}{r^{2k+2+2l+2}(2k+1)(2l+1)}\Big(\underbrace{\cos((2k+1)\varphi)\cos((2l+1)\varphi)}_{\pi\delta_{k,l}}+\underbrace{\sin((2k+1)\varphi)\sin((2l+1)\varphi)}_{\pi\delta_{k,l}}\Big)r\,dr\,d\varphi$$

$$=\frac{2\mu_0 i^2}{\pi^2}2\pi\left(\sum_{k=0}^\infty\int_0^R\frac{r^{4k+1}}{R^{4k}(2k+1)^2}dr+\sum_{k=0}^\infty\int_R^\infty\frac{R^{4k+4}}{r^{4k+3}(2k+1)^2}dr\right)$$

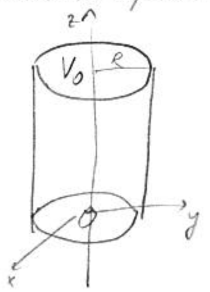
$$=\frac{4\mu_0 i^2}{\pi}\left(\sum_{k=0}^\infty\frac{R^{4k+2}}{(4k+2)R^{4k}(2k+1)^2}+\sum_{k=0}^\infty\frac{R^{4k+4}}{(2k+1)^2(4k+2)}\frac{1}{R^{4k+2}}\right)=\frac{4\mu_0 i^2R^2}{\pi}\sum_{k=0}^\infty\frac{1}{(2k+1)^3}$$

$$\zeta(3)=\sum_{k=1}^\infty\frac{1}{k^3}=\sum_{k=0}^\infty\frac{1}{(2k+1)^3}+\sum_{k=1}^\infty\frac{1}{(2k)^3}=\sum_{k=0}^\infty\frac{1}{(2k+1)^3}+\frac{1}{8}\sum_{k=0}^\infty\frac{1}{k^3}=\frac{1}{8}\zeta(3)+\sum_{k=0}^\infty\frac{1}{(2k+1)^3}\Rightarrow$$

$$\sum_{k=0}^\infty\frac{1}{(2k+1)^3}=\frac{7}{8}\zeta(3)$$

$$\Rightarrow\frac{dW}{dz}=\frac{4\mu_0 i^2R^2}{\pi}\frac{7}{8}\zeta(3)=\boxed{\frac{7\mu_0 i^2R^2}{2\pi}\zeta(3)}$$

Одредити потенцијал електростатичког поља у унутрашњости цилиндра полупречника R и висине h, ако су једна основа и омотач на нултом потенцијалу, а друга основа је на константном потенцијалу V_0. У унутрашњости цилиндра нема наелектрисања.



$\Delta V=0$ → очигледно $V=V(r,z)$ али узимамо и φ зависност педагошки!

$$\Delta V=\frac{\partial^2V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2V}{\partial\varphi^2}+\frac{\partial^2V}{\partial z^2}\quad;\ V=R(r)P(\varphi)Q(z)$$

$$R''PQ+\frac{1}{r}R'PQ+\frac{1}{r^2}RP'Q+RPQ''=0\quad\Big/\frac{1}{PQR}$$

$$\underbrace{\frac{1}{R}R''+\frac{1}{rR}R'+\frac{1}{r^2}\frac{1}{P}P'}_{f(r,\varphi)=-k^2}+\underbrace{\frac{1}{Q}Q''}_{f(z)=k^2}=0$$

$f(z)=k^2$ ← није периодично по z

→ $Q(z)\sim e^{-kz}$, e^{kz} или $sh(kz)$ и $ch(kz)$

$Q(z=0)=0\Rightarrow\quad Q(z)\sim sh(kz)$

$$\frac{R''}{R}+\frac{R'}{rR}+\frac{1}{r^2}\frac{P'}{P}=-k^2\ \Big/r^2\qquad\underbrace{r^2\frac{R''}{R}+r\frac{R'}{R}+k^2r^2}_{g(r)}=\underbrace{-\frac{P'}{P}}_{g(\varphi)}=m^2$$

по φ периодично!

$P(\varphi)\sim\cos m\varphi,\ \sin m\varphi,\quad m\in\mathbb{N}_0$ укључим и 0 јер је то константно решење.
(за $m=0$ решење по φ $\sim \underset{0}{\cancel{A}}\varphi+B$)

$$r^2\frac{R''}{R}+r\frac{R'}{R}+k^2r^2=m^2 \;\Big/\;\frac{R}{r^2} \Rightarrow \boxed{R''+\frac{1}{r}R'+\left(k^2-\frac{m^2}{r^2}\right)R=0}$$ смена $rk=x$

$$\boxed{\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}+\left(1-\frac{m^2}{x^2}\right)R(x)=0}$$ Беселова диф ј-на

за $m\notin\mathbb{Z}$ решења J_m и J_{-m}, за $m\in\mathbb{Z}$ J_m и N_m дивергира у $x=0$

$\Rightarrow R(x)\sim J_m(x)$

Гранични услов $V(r=R)=0$ омотач је на нултом потенцијалу

$\Rightarrow J_m(k\cdot R)=0 \Rightarrow k\cdot R=x_{mn}$ нуле Беселове ф-је реда m; $n=1,2,\dots$

$\rightarrow \boxed{k=\frac{x_{mn}}{R}}$

$$\Rightarrow V(r,\varphi,z)=\underbrace{\sum_{n=1}^{\infty}}_{\text{броји нуле}}\sum_{m=0}^{\infty}\left(A_{mn}\sin m\varphi+B_{mn}\cos m\varphi\right)\mathrm{sh}\left(\frac{x_{mn}}{R}z\right)J_m\left(\frac{x_{mn}}{R}r\right)$$

Случај $\underline{k=0}$ није размотрен $\Theta\sim C_1z+C_2$ по R: $R''+\frac{1}{r}R'-\frac{m^2}{r^2}R=0$ (Ојлер. диф. ј.)

$\Rightarrow R\sim r^{\alpha}\quad \alpha(\alpha-\cancel{1})+\cancel{\alpha}-m^2=0 \Rightarrow R\sim r^m, \cancel{\frac{1}{r^m}}$ дивергира у 0.

$$V(R,\varphi,z)=0 \Rightarrow \sum_{m=0}^{\infty}\left(A_m\sin m\varphi+B_m\cos m\varphi\right)R^m(z+C)\;\Big/\int_0^{2\pi}\sin n\varphi\,d\varphi$$

$A_n\pi R^n(z+C)=0 \Rightarrow \boxed{A_n=0}\ \forall n$

$B_n\pi R^n(z+C)=0 \Rightarrow \boxed{B_n=0}\ n\neq 0$ за $m=0$ $B_0\cdot(z+C)=0\Rightarrow\boxed{B_0=0}$

Значи, $k=0$ не даје ништа ново!

Последњи гр. услов:

$$V(r,\varphi,z=h)=V_0=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left(A_{mn}\sin m\varphi+B_{mn}\cos m\varphi\right)\mathrm{sh}\left(\frac{x_{mn}}{R}h\right)J_m\left(\frac{x_{mn}}{R}r\right)\;\Big/\int_0^{2\pi}\sin(m'\varphi)d\varphi\;\Big/\int_0^{2\pi}\cos(m'\varphi)d\varphi\quad m'\neq0$$

$$V_0\int_0^{2\pi}\sin(m'\varphi)d\varphi=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}A_{mn}\,\pi\,\delta_{m,m'}\,\mathrm{sh}\left(\frac{x_{mn}}{R}h\right)J_m\left(\frac{x_{mn}}{R}r\right)$$

$$-V_0\cos(m'\varphi)\Big|_0^{2\pi}=0=\sum_{n=1}^{\infty}A_{m'n}\,\pi\,\mathrm{sh}\left(\frac{x_{m'n}}{R}h\right)\underbrace{J_{m'}\left(\frac{x_{m'n}}{R}r\right)}_{\text{ортог. ф-је базис у } r \text{ простору!}} \Rightarrow \underline{A_{m'n}=0}$$

$$V_0\int_0^{2\pi}\cos(m'\varphi)d\varphi=0=\sum_{n=1}^{\infty}B_{m'n}\,\pi\,\mathrm{sh}\left(\frac{x_{m'n}}{R}h\right)J_{m'}\left(\frac{x_{m'n}}{R}r\right) \Rightarrow B_{m'n}=0 \text{ за } m'\neq0$$

остаје само члан $\underline{m=0}$ што смо у старту знали! Нема φ зависности.

$$V(r,\varphi,z)=\sum_{n=1}^{\infty}B_{0n}\,\mathrm{sh}\left(\frac{x_{0n}}{R}z\right)J_0\left(\frac{x_{0n}}{R}r\right)$$

$$V_0=\sum_{n=1}^{\infty}B_{0n}\,\mathrm{sh}\left(\frac{x_{0n}}{R}h\right)J_0\left(\frac{x_{0n}}{R}r\right)\;\Big/\int_0^R r\,dr\,J_0\left(\frac{x_{0n'}}{R}r\right)$$

$$V_0 \int_0^R r\,dr\, \underbrace{J_0\left(\frac{x_{0n'}}{R} r\right)}_{\sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,j!}\left(\frac{x_{0n'} r}{2R}\right)^{2j}} = \sum_{n=1}^{\infty} B_{0n}\, sh\left(\frac{x_{0n}}{R} h\right) \underbrace{\int_0^R J_0\left(\frac{x_{0n}}{R} r\right) J_0\left(\frac{x_{0n'}}{R} r\right) r\,dr}_{\frac{R^2}{2} J_1^2(x_{0n'})\,\delta_{nn'}}$$

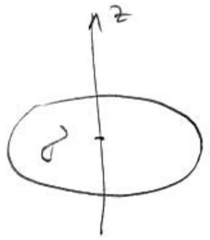
$$V_0 \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,j!} \frac{x_{0n'}^{2j}}{(2R)^{2j}} \underbrace{\int_0^R r^{2j+1} dr}_{\frac{R^{2j+2}}{2j+2}} = \frac{R^2}{2} J_1^2(x_{0n'})\, B_{0n'}\, sh\left(\frac{x_{0n'}}{R} h\right)$$

$$V_0 \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!}\left(\frac{x_{0n'}}{2R}\right)^{2j} \frac{R^{2j+2}}{2} = \frac{R^2}{2} J_1^2(x_{0n'})\, B_{0n'}\, sh\left(\frac{x_{0n'}}{R} h\right)$$

$$V_0 \cdot \frac{2}{x_{0n'}} \cancel{J_1(x_{0n'})} = J_1^{\cancel{2}}(x_{0n'})\, sh\left(\frac{x_{0n'}}{R} h\right) B_{0n'} \Rightarrow B_{0n'} = \frac{2V_0}{x_{0n'} J_1(x_{0n'})\, sh\left(\frac{x_{0n'}}{R} h\right)}$$

$$\Rightarrow \boxed{V(r,z) = \sum_{n=1}^{\infty} \frac{2V_0\, sh\left(\frac{x_{0n} z}{R}\right) J_0\left(\frac{x_{0n}}{R} r\right)}{sh\left(\frac{x_{0n} h}{R}\right) J_1(x_{0n})\, x_{0n}}}$$

Танак диск полупречника R је наелектрисан површинском густином наелектрисања σ. Користећи развој потенцијала у Фурије-Беселов интеграл одредити потенцијал V у целом простору.



решавамо $\Delta V = -\frac{1}{\varepsilon_0}\rho = 0$ изнад и испод z-осе!

$$V = V(r,z) = R(r)Q(z) \qquad \Delta V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2}$$

$$R''Q + \frac{1}{r}R'Q + RQ'' = 0 \quad \Big/ \frac{1}{RQ}$$

$$\frac{1}{R}R'' + \frac{1}{rR}R' + \underbrace{\frac{1}{Q}Q''}_{K^2} = 0$$ хоћемо експ. опадање по z-оси!

$\underline{k>0}$ $\quad Q \sim e^{-kz}, e^{kz}$ $\quad$ за $z>0$ узимамо e^{-kz}, за $z<0$ e^{kz}

$$\Rightarrow Q \sim e^{-k|z|}$$

$$\frac{1}{R}R'' + \frac{1}{rR}R' + k^2 = 0 \Rightarrow \quad R'' + \frac{1}{r}R' + k^2 R = 0$$ Беселова јна смене $x = k \cdot r$

$$\boxed{\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + R = 0}$$ Беселова јна $\to$ решење $J_0(kr)$ и $\cancel{N_0}(kr)$ див. у $r=0$

$$\Rightarrow V \sim e^{-k|z|} J_0(kr)$$ интегралимо по свим k!

$$V(z,r) = \int_0^{\infty} b_\lambda J_0(\lambda r)\, e^{-\lambda |z|}\, d\lambda$$

гр. услов: $E_{>n} - E_{<n} = \begin{cases} \sigma/\varepsilon_0, & r \le R \\ 0, & r > R \end{cases}$

$$-\frac{\partial V_>}{\partial z} + \frac{\partial V_<}{\partial z}\Big|_{z=0} = \int_0^\infty \lambda b_\lambda J_0(\lambda r)\,d\lambda + \int_0^\infty \lambda b_\lambda J_0(\lambda r)\,d\lambda = 2\int_0^\infty \lambda b_\lambda J_0(\lambda r)\,d\lambda = \frac{\sigma}{\varepsilon_0}\,\eta(R-r)$$

$$\Big/ \int_0^\infty J_0(\lambda' r)\, r\,dr$$

$$2\int_0^\infty \lambda b_\lambda d\lambda \underbrace{\int_0^\infty r\,dr\, J_0(\lambda r) J_0(\lambda' r)} = \frac{\sigma}{\varepsilon_0}\int_0^R J_0(\lambda' r)\,dr \cdot r$$

$$2\int_0^\infty \cancel{\lambda} b_\lambda d\lambda \; \frac{1}{\cancel{\lambda}}\delta(\lambda-\lambda') = \frac{\sigma}{\varepsilon_0}\int_0^R \sum_{j=0}^\infty \frac{(-1)^j}{j!\,j!}\left(\frac{\lambda' r}{2}\right)^{2j} r\,dr$$

$$2 b_{\lambda'} = \frac{\sigma}{\varepsilon_0}\sum_{j=0}^\infty \frac{(-1)^j}{j!\,j!}\left(\frac{\lambda'}{2}\right)^{2j}\frac{R^{2j+2}}{2j+2} = \frac{\sigma}{2\varepsilon_0}R^2\sum_{j=0}^\infty \frac{(-1)^j}{j!\,(j+1)!}\left(\frac{\lambda' R}{2}\right)^{2j} = \frac{\sigma}{\cancel{2}\varepsilon_0}R^{\cancel{2}}\frac{\cancel{2}}{\lambda'\cancel{R}}J_1(\lambda' R)$$

$$\Rightarrow b_{\lambda'} = \frac{\sigma R}{2\varepsilon_0\lambda'}J_1(\lambda' R)$$

$$\Rightarrow \boxed{V(r,z) = \frac{\sigma R}{2\varepsilon_0}\int_0^\infty \frac{1}{\lambda}J_1(\lambda R)\cdot J_0(\lambda r)\,e^{-\lambda|z|}\,d\lambda}$$